The isoperimetric problem for the ground state energy of a Schrödinger operator

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Introduction – Functional inequalities

Sharp constants → **Characterization of optimizers** → **Stability bounds**

The **quantitative isoperimetric inequality** of **Fusco–Maggi–Pratelli** (2008) (answering a question of Hall (1992) and extending earlier work starting with Bonnesen (1924))

$$\frac{\mathrm{Per}(\Omega)}{|\Omega|^{(d-1)/d}} \geq d\omega_d^{1/d} + c_d \inf_{a\in\mathbb{R}^d, r>0} \frac{|\Omega \Delta (rB+a)|^2}{|rB|^2}\,.$$

The **quantitative Sobolev inequality** of **Bianchi–Egnell** (1991) (answering a question of Brezis–Lieb (1985))

$$\|\nabla\psi\|^2 \geq S_d \|\psi\|_{2d/(d-2)}^2 + c_d' \inf_{Q\in\mathcal{M}} \|\nabla(\psi - Q)\|^2\,.$$

where $\mathcal{M} = \{ch(b(\cdot - a)) :\ c\in\mathbb{R}\,, b>0\,, a\in\mathbb{R}^d\}$ is the manifold of optimizers
Bianchi–Egnell introduced **compactness + linearization method**

Further results on **Faber–Krahn**, **Szegő–Weinberger**, **Brunn–Minkowski**, ...

The isoperimetric problem for Schrödinger operators

Problem (**Keller** (1961)): Given $1 \leq p < \infty$ and $m > 0$, how small can

$$\lambda_1(-\Delta + V) = \inf_\psi \frac{\int_{\mathbb{R}^d} \left(|\nabla\psi|^2 + V|\psi|^2\right) dx}{\int_{\mathbb{R}^d} |\psi|^2 \, dx}$$

be under the constraint $\int_{\mathbb{R}^d} |V|^p \, dx = m$?

Properties: • By Sobolev inequalities, answer is $-\infty$ if $p = 1$ in $d = 2$, $p < d/2$ in $d \geq 3$. Answer for $p = d/2$ in $d \geq 3$ depends on m. **From now** on $p = \gamma + d/2 > d/2$.

• Positive and **negative** parts of V play a different role. May assume $V \leq 0$.

• By **scale covariance**, this problem is equivalent to computing

$$\mathcal{C}_{\gamma,d} = \inf_V \frac{\lambda_1(-\Delta + V)}{\left(\int_{\mathbb{R}^d} V_-^{\gamma+d/2} \, dx\right)^{1/\gamma}} .$$

• Problem is **translation invariant** and **rotation invariant**.

• Keller suggests an **explicit solution** in $d = 1$, namely, $V(x) = -c_\gamma \cosh^{-2}(x)$.

Main result I

Recall

$$\mathcal{C}_{\gamma,d} = \inf_V \frac{\lambda_1(-\Delta+V)}{\left(\int_{\mathbb{R}^d} V_-^{\gamma+d/2}\,dx\right)^{1/\gamma}}\,.$$

Theorem 1 (**Existence and uniqueness of an optimal potential**). *Let $\gamma > 1/2$ if $d=1$ and $\gamma>0$ if $d\geq 2$.* **There is** *a non-positive function $\mathcal{V}$, which is* **unique** *up to translations and dilations, such that*

$$\mathcal{M} := \left\{ V\in L^{\gamma+d/2}(\mathbb{R}^d):\ \lambda_1(-\Delta+V) = \mathcal{C}_{\gamma,d}\left(\int_{\mathbb{R}^d} V_-^{\gamma+d/2}\,dx\right)^{1/\gamma}\right\}$$
$$= \left\{ b^2\mathcal{V}(b(\cdot-a)):\ b>0\,, a\in\mathbb{R}^d\right\}\,.$$

Remark. The functions $\mathcal{V}$ are known explicitly only in 1D.

Main result II

Theorem 2 (**Stability**)**.** *Let $\gamma > 1/2$ if $d = 1$ and $\gamma > 0$ if $d \geq 2$. Then:*

(i) For $\gamma + d/2 \leq 2$, there is a constant $c_{\gamma,d} > 0$ such that for any $V \in L^{\gamma+d/2}(\mathbb{R}^d)$,

$$\frac{\lambda_1(-\Delta + V)}{\left(\int_{\mathbb{R}^d} V_-^{\gamma+d/2}\, dx\right)^{1/\gamma}} \geq \mathcal{C}_{\gamma,d} + c_{\gamma,d} \inf_{W \in \mathcal{M}} \frac{\|V_- - W_-\|_{\gamma+d/2}^2}{\|V_-\|_{\gamma+d/2}^2}\,.$$

(ii) For $\gamma + d/2 \geq 2$, there is a constant $c_{\gamma,d} > 0$ such that for any $V \in L^{\gamma+d/2}(\mathbb{R}^d)$,

$$\frac{\lambda_1(-\Delta + V)}{\left(\int_{\mathbb{R}^d} V_-^{\gamma+d/2}\, dx\right)^{1/\gamma}} \geq \mathcal{C}_{\gamma,d} + c_{\gamma,d} \inf_{W \in \mathcal{M}} \frac{\left\| V_-^{2/(q-2)} - W_-^{2/(q-2)} \right\|_{q/2}^2}{\left\| V_-^{2/(q-2)} \right\|_{q/2}^2}\,,$$

where q is related to γ and d by $1/(\gamma + d/2) + 2/q = 1$.

Interchanging the infima

Let again $1/(\gamma + d/2) + 2/q = 1$ and, wlog, assume that $\int V_-^{\gamma+d/2}\,dx = 1$.

$$\int_{\mathbb{R}^d} \left(|\nabla\psi|^2 + V|\psi|^2\right) dx = \underbrace{\int_{\mathbb{R}^d} |\nabla\psi|^2\,dx - \left(\int_{\mathbb{R}^d} |\psi|^q dx\right)^{2/q}}_{=:\mathcal{A}}$$

$$+ \underbrace{\left(\int_{\mathbb{R}^d} |\psi|^q dx\right)^{2/q} - \int_{\mathbb{R}^d} V_-|\psi|^2\,dx}_{=:\mathcal{B}} + \underbrace{\int_{\mathbb{R}^d} V_+|\psi|^2\,dx}_{=:\mathcal{C}}\,.$$

- Clearly, $\mathcal{C} \geq 0$.
- By **Hölder**, $\mathcal{B} \geq 0$ with equality iff $V_- = \|\psi\|_q^{(2-q)/q}|\psi|^{q-2}$.
- By a **Gagliardo–Nirenberg–Sobolev** (GNS) inequality, $\mathcal{A} \geq -C'_{q,d}\|\psi\|^2$.

This argument (essentially due to **Lieb–Thirring** (1976)) shows that

$$\mathcal{C}_{\gamma,d} = \inf_V \frac{\lambda_1(-\Delta+V)}{\left(\int_{\mathbb{R}^d} V_-^{\gamma+d/2}\,dx\right)^{1/\gamma}} = \inf_\psi \frac{\int_{\mathbb{R}^d}|\nabla\psi|^2\,dx - \left(\int_{\mathbb{R}^d}|\psi|^q dx\right)^{2/q}}{\int_{\mathbb{R}^d}|\psi|^2\,dx} = \mathcal{C}'_{q,d}\,.$$

Strategy of our proof

Recall

$$\int_{\mathbb{R}^d} \left(|\nabla\psi|^2 + V|\psi|^2\right) dx = \underbrace{\int_{\mathbb{R}^d} |\nabla\psi|^2 \, dx - \left(\int_{\mathbb{R}^d} |\psi|^q dx\right)^{2/q}}_{=:\mathcal{A}}$$

$$+ \underbrace{\left(\int_{\mathbb{R}^d} |\psi|^q dx\right)^{2/q} - \int_{\mathbb{R}^d} V_-|\psi|^2 \, dx}_{=:\mathcal{B}} + \underbrace{\int_{\mathbb{R}^d} V_+|\psi|^2 \, dx}_{=:\mathcal{C}} .$$

Existence and uniqueness of an optimal V in the isoperimetric problem (**Theorem 1**) follows from the existence and uniqueness of an optimal ψ for GNS.

Stability in the isoperimetric problem (**Theorem 2**) follows from **both** a stability result for Hölder and for GNS.

This is what we will discuss below.

Hölder's inequality with remainder

For $f \in L^p(X,\mu)$ define the **duality map**

$$\mathcal{D}_p(f) = \|f\|_p^{1-p} |f|^{p-2} \overline{f} \,.$$

Theorem 3. *Let $p \geq 2$. Let $f \in L^p(X,\mu)$ and $g \in L^{p'}(X,\mu)$ with $\|f\|_p = \|g\|_{p'} = 1$. Then*

$$\left| \int_X fg \, \mathrm{d}\mu \right| \leq 1 - \frac{p'-1}{4} \|\mathcal{D}_p(f) - e^{i\theta} g\|_{p'}^2 \ ,$$

and

$$\left| \int_X fg \, \mathrm{d}\mu \right| \leq 1 - \frac{1}{p \ 2^{p-1}} \|e^{i\theta} f - \mathcal{D}_{p'}(g)\|_p^p \ .$$

where $\theta \in [0, 2\pi)$ is such that $e^{i\theta} \int_X fg \, \mathrm{d}\mu$ is non-negative. The exponents 2 and p on the right sides are best possible.

Proof uses **uniform convexity** of L^p in the form that for unit vectors u and v,

$$\left\| \frac{u+v}{2} \right\|_{p'} \leq 1 - \frac{p'-1}{8} \|u-v\|_{p'}^2 \ , \qquad \left\| \frac{u+v}{2} \right\|_p \leq 1 - \frac{1}{p \ 2^p} \|u-v\|_p^p \,.$$

GNS inequality with remainder

Set $\mathcal{E}_q[\psi] = \int_{\mathbb{R}^d} |\nabla\psi|^2\,dx - \left(\int_{\mathbb{R}^d} |\psi|^q\,dx\right)^{2/q}$ and recall that

$$\mathcal{C}'_{q,d} = \inf\{\mathcal{E}_q[\psi] :\ \|\psi\| = 1\}\,.$$

Theorem 4. *Let $2 < q < \infty$ if $d = 1, 2$ and $2 < q < 2d/(d-2)$ if $d \geq 3$.* **There is** *a function Q, which is* **unique** *up to translations and a sign, such that*

$$\begin{aligned}\mathcal{G} &:= \left\{\psi \in H^1(\mathbb{R}^d) :\ \mathcal{E}_q[\psi] = \mathcal{C}'_{q,d}\,, \|\psi\|^2 = 1\right\} \\ &= \left\{\sigma Q(\cdot - a) :\ a \in \mathbb{R}^d,\ \sigma = \pm 1\right\}\,.\end{aligned}$$

Theorem 5. *Let $2 < q < \infty$ if $d = 1, 2$ and $2 < q < 2d/(d-2)$ if $d \geq 3$. Then there is a constant $c'_{q,d} > 0$ such that for all $\psi \in H^1(\mathbb{R}^d)$ with $\|\psi\| = 1$*

$$\mathcal{E}_q[\psi] \geq \mathcal{C}'_{q,d} + c'_{q,d} \inf_{\phi\in\mathcal{G}} \|\psi - \phi\|^2_{H^1}\,.$$

Remark. Our $c'_{q,d}$ comes via **compactness** and is not explicit. Dolbeault–Toscani (Preprint 2012) have stability for a different GNS inequality, but with explicit constants.

GNS inequality with remainder, cont'd

Finally, need to bound the remainder $\|\psi - \phi\|_{H^1}^2$ from below.

Lemma 6. *Let $f, g \in L^q(X,\mu)$ Then for all $q \geq 2$,*

$$\|f - g\|_q \geq \frac{1}{4} \max\{\, \|f\|_q \ , \ \|g\|_q\} \left\| \frac{|f|^2}{\|f\|_q^2} - \frac{|g|^2}{\|g\|_q^2} \right\|_{q/2} .$$

Moreover, if $q \geq 4$, then

$$\|f - g\|_q \geq \frac{1}{4(q-2)} \max\{\, \|f\|_q \ , \ \|g\|_q\} \left\| \frac{|f|^{q-2}}{\|f\|_q^{q-2}} - \frac{|g|^{q-2}}{\|g\|_q^{q-2}} \right\|_{q/(q-2)} .$$

Ingredients in the proofs of Theorems 4 and 5:

- **Existence** of a minimizer, possible loss of compactness due to translations, Lieb (1983)
- Every positive solution of the Euler–Lagrange equation is **radial** (MMP or strict Riesz).
- **Kwong's theorem I** (1989): There is a **unique** positive radial solution vanishing at infinity.
- **Kwong's theorem II** (1989): This solution is **non-degenerate**, i.e., the linearization of the equation around the solution has only the trivial zero modes due to translation.

THANK YOU FOR YOUR ATTENTION!