Recent advances in isoperimetric inequalities for eigenvalues

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Outline

Introduction

Dirichlet eigenvalues
   Minimization of $\lambda_k(\Omega)$
   Some other problems

Neumann eigenvalues
   Maximization of $\mu_k$
   Numerical results

Robin eigenvalues

Steklov eigenvalues
   Maximization of $p_k$
   Other inequalities
   The trace operator
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The trace operator
Notations

**DIRICHLET:** $\lambda_k(\Omega)$

\[
\begin{aligned}
-\Delta u &= \lambda u \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial\Omega
\end{aligned}
\]

**ROBIN:** $\sigma_k(\Omega, \alpha)$

\[
\begin{aligned}
-\Delta u &= \sigma u \quad \text{in } \Omega \\
\frac{\partial u}{\partial n} + \alpha u &= 0 \quad \text{on } \partial\Omega
\end{aligned}
\]

**NEUMANN:** $\mu_k(\Omega)$ ($\mu_0 = 0$)

\[
\begin{aligned}
-\Delta u &= \mu u \quad \text{in } \Omega \\
\frac{\partial u}{\partial n} &= 0 \quad \text{on } \partial\Omega
\end{aligned}
\]

**STEKLOV:** $p_k(\Omega)$ ($p_0 = 0$)

\[
\begin{aligned}
\Delta u &= 0 \quad \text{in } \Omega \\
\frac{\partial u}{\partial n} &= pu \quad \text{on } \partial\Omega
\end{aligned}
\]
Isoperimetric inequalities

We want to prove isoperimetric inequalities or optimal bounds for the eigenvalues or some functions of the eigenvalues. These bounds will usually depend on geometric quantities like the volume $|\Omega|$, the perimeter $P(\Omega)$ or the diameter $D(\Omega)$. 
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Therefore, we will consider problems like
\[
\min \{ \lambda_k(\Omega); \Omega \in \mathbb{R}^N; |\Omega| = c \}, \min \{ \lambda_k(\Omega); \Omega \in \mathbb{R}^N; P(\Omega) = c \},
\]
etc...
Isoperimetric inequalities

We want to prove isoperimetric inequalities or optimal bounds for the eigenvalues or some functions of the eigenvalues. These bounds will usually depend on geometric quantities like the volume $|\Omega|$, the perimeter $P(\Omega)$ or the diameter $D(\Omega)$.

Therefore, we will consider problems like

$$\min\{\lambda_k(\Omega); \Omega \in \mathbb{R}^N; |\Omega| = c\}, \min\{\lambda_k(\Omega); \Omega \in \mathbb{R}^N; P(\Omega) = c\},$$

etc...

By homogeneity, it is equivalent to consider problems like

$$\min\{|\Omega|^{2/N} \lambda_k(\Omega); \Omega \in \mathbb{R}^N\}, \min\{P(\Omega)^{2/(N-1)} \lambda_k(\Omega); \Omega \in \mathbb{R}^N\},$$

etc...
## The two lowest eigenvalues (volume constraint)

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<td>Weinstock 1954</td>
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<td>Weinberger 1956</td>
<td>Daners 2006</td>
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<td><strong>2nd eigenvalue</strong></td>
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**Steklov eigenvalues**
- Maximization of $p_k$
- Other inequalities
- The trace operator
A general existence result

Theorem (Bucur; Mazzoleni-Pratelli 2011)

The problem \( \min\{\lambda_k(\Omega), \Omega \subset \mathbb{R}^N, |\Omega| = c\} \) has a solution. This one is an open set which is bounded and has finite perimeter.

Open problems**: what is the regularity of the minimizers? Are they connected? Simply connected? What are the symmetries?

Open problem*: If \( \Omega^* \) denotes a minimizer for \( \lambda_k, k \geq 2 \), prove that \( \lambda_k \) is a multiple eigenvalue, \( \lambda_k - 1(\Omega^*) = \lambda_k(\Omega^*) \).
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More generally, the problem
\[
\min\{F(\lambda_1(\Omega), \ldots, \lambda_p(\Omega)), \Omega \subset \mathbb{R}^N, |\Omega| = c\},
\]
where \( F : \mathbb{R}^p \to \mathbb{R} \) is increasing in each variable and lower-semicontinuous, has a solution.
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Techniques of proof

The authors use two different techniques:

Mazzoleni-Pratelli: they are able to replace any minimizing sequence by a uniformly bounded one and then apply Buttazzo-DalMaso Theorem.
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Mazzoleni-Pratelli: they are able to replace any minimizing sequence by a uniformly bounded one and then apply Buttazzo-DalMaso Theorem.

Bucur introduces the notion of *local shape sub-solution for the energy* (which are bounded), proves that minimizers for the eigenvalues satisfy this definition and conclude by induction thanks to a concentration-compactness argument.
The third eigenvalue $\lambda_3$

Dimension 2:
Open problem*** Prove that the disk is the minimizer!
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$\triangleright$ the disk is a local minimizer for $\lambda_3$ (Wolf-Keller 1994)
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Known:

- the disk is a local minimizer for $\lambda_3$ (Wolf-Keller 1994)
- $\lambda_1$ and $\lambda_3$ are the only eigenvalues for which the disk is a local minimizer (A. Berger 2013)

**Dimension 3:** the ball is not the minimizer (numerical evidence Oudet 2010, proof by A. Berger 2013).

**Dimension $\geq 4$:** Open problem** Prove that the union of three identical balls is the minimizer.
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Numerical results for $\lambda_k$

Numerical results for the minimization of $\lambda_k(\Omega)$, $k = 4 \ldots 15$ have been obtained in the plane e.g. by E. Oudet (2004), P. Antunes and P. Freitas (2012).
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Table: Minimizers of $\lambda_k(\Omega)$, $k = 4 \ldots 15$ in the plane, by courtesy of P. Antunes and P. Freitas
Symmetry?

P. Antunes and P. Freitas got, as a possible solution for the minimizer of $\lambda_{13}$ the following domain

To confirm this non-symmetry result, they tried to look for the best symmetric domain (by imposing an axis of symmetry) and they got the following domain
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But the second one has a worse 13-th eigenvalue than the first one. Thus it may appear that the minimizers are not necessarily symmetric.
Some three-dimensional results

Table: Minimizers of $\lambda_k(\Omega)$, $k = 2 \ldots 10$ in the 3D space, by courtesy of A. Berger and E. Oudet
Perimeter constraint

Theorem (Bucur-Buttazzo-H. 2009; De Philippis-Velichkov 2013)

The problem \( \min \{ \lambda_k(\Omega), \Omega \subset \mathbb{R}^N, P(\Omega) = c \} \) has a solution. This one is bounded, connected. Its boundary is \( C^{1,\alpha} \) outside a closed set of Hausdorff dimension at most \( N - 8 \). It is analytic in dimension \( N = 2 \).
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The minimizer for \( \lambda_2 \) in the plane is a regular strictly convex domain with a curvature vanishing at exactly two points.
First motivation: the gap conjecture (which is now the gap theorem by B. Andrews and J. Clutterbuck!). We wanted to prove (see AIM Palo-Alto meeting Low Eigenvalues of Laplace and Schrödinger Operators in 2006) that the problem

$$\min\{\lambda_2(\Omega) - \lambda_1(\Omega); \Omega \text{ convex } ; D(\Omega) = 1\}$$

has no solution.
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Then we were led to the problems of minimizing $\lambda_2(\Omega) - k\lambda_1(\Omega)$, $0 \leq k \leq 1$ and $\lambda_2(\Omega)$ among convex domains with fixed diameter.

Theorem

The problem

$$\min\{\lambda_k(\Omega), \Omega \subset \mathbb{R}^N, D(\Omega) = c\}$$

has a solution. This one is a convex domain of constant width. The minimizer for $\lambda_1$ is obviously the ball.

Open problem**: prove that the ball minimizes the second eigenvalue with a diameter constraint.

Is it possible that the ball is the minimizer for any $\lambda_k$ with a diameter constraint?
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The eigenvalues of

\[\begin{aligned}
-\Delta u &= \mu u \quad \text{in } \Omega \\
\frac{\partial u}{\partial n} &= 0 \quad \text{on } \partial \Omega
\end{aligned}\]

are \(\mu_0 = 0 \leq \mu_1 \leq \mu_2 \ldots\)

Open problem**: Prove a general existence result for \(\max\{\mu_k(\Omega); \Omega \subset \mathbb{R}^N, \Omega \text{ bounded and Lipschitz}, |\Omega| = c\} \).
The eigenvalues of
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The second Neumann eigenvalue

Theorem (A. Girouard-N. Nadirashvili-I. Polterovitch 2009)

The union of two disjoint balls solves the problem

\[ \max\{\mu_2(\Omega), \Omega \subset \mathbb{R}^2, \Omega \text{ regular, simply connected}, |\Omega| = c\} . \]
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Idea of the proof: folding and rearrangement method, together with conformal maps and a topological argument.
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It remains to prove:

Open problem**: Extend the theorem to non simply-connected domains and to higher dimensions.
Numerical results

Numerical results for the maximization of $\mu_k(\Omega)$, $k = 4 \ldots 15$ have been obtained in the plane e.g. by P. Antunes and P. Freitas (2012), A. Berger and E. Oudet (2013)
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**Table:** Maximizers of $\mu_k(\Omega)$, $k = 2 \ldots 10$ in the plane, by courtesy of A. Berger and E. Oudet
Numerical results - 3D case

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No general existence result available. The case of $\sigma_1(\Omega, \alpha)$ has been solved in 1986 by M.H. Bossel in the plane and more recently in 2006 by D. Daners in any dimension, following the same strategy.

Theorem (J. Kennedy 2009)

The union of two disjoint balls solves the problem

$$\min\{\sigma_2(\Omega, \alpha), \Omega \subset \mathbb{R}^N, \Omega \text{ bounded and Lipschitz}, |\Omega| = c\}$$

for any $\alpha > 0$.

Idea of the proof: similar to the Dirichlet case. Some supplementary work to deal with the possible non regularity of the nodal surface.

The case $\alpha < 0$ seems completely open even for $\sigma_1(\Omega, \alpha)$.

Open problem**: prove that the ball maximizes $\sigma_1(\Omega, \alpha)$ for $\alpha < 0$ among bounded Lipschitz domains.
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are \( p_0 = 0 \leq p_1 \leq p_2 \ldots \).... We can look at problems like

\[
\max \{ p_k(\Omega); \Omega \subset \mathbb{R}^N, \Omega \text{ bounded and Lipschitz} \}
\]

either with an area constraint \( |\Omega| = c \) or a perimeter constraint \( P(\Omega) = c \).

Theorem

▶ **Weinstock 1954:** if \( N = 2 \), the disk maximizes \( p_1(\Omega) \) among sets of given perimeter.

▶ **Brock 2001:** if \( N \geq 2 \), the ball maximizes \( p_1(\Omega) \) among sets of given volume.

Open problem**: Extend Brock’s result to the perimeter constraint.
The second Steklov eigenvalue

Theorem (A. Girouard-I. Polterovitch 2009)

The union of two disjoint disks solves the problem

$$\max\{p_2(\Omega), \Omega \subset \mathbb{R}^2, \Omega \text{ regular and simply connected}, P(\Omega) = c\}.$$
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Idea of the proof: similar to the Neumann case (conformal map plus a topological argument)

Open problem**: Extend the Theorem to non simply-connected domains and to higher dimensions.
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It remains to prove:

Open problem**: Extend the Theorem to non simply-connected domains and to higher dimensions.
Some other inequalities

Let $\Omega$ be any (regular) domain and denote by $\Omega^*$ the ball with same volume. F. Brock in 2001 proved the inequality:

$$
\sum_{i=2}^{N+1} \frac{1}{p_i(\Omega)} \geq \sum_{i=2}^{N+1} \frac{1}{p_i(\Omega^*)}
$$

which clearly implies $p_2(\Omega) \leq p_2(\Omega^*)$. 

Theorem (H.-Philippin-Safoui 2008)

For any convex domain in $\mathbb{R}^N$, we have

$$
\Pi_{N+1} \sum_{k=2}^{N+1} p_k(\Omega) \leq \Pi_{N+1} \sum_{k=2}^{N+1} p_k(\Omega^*)
$$

Open problem*: remove the convexity assumption in the previous inequality
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In two-dimensions, Hersch-Payne-Schiffer proved in 1975 a stronger inequality

$$p_2(\Omega)p_3(\Omega) \leq p_2(\Omega^*)p_3(\Omega^*)$$
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**Open problem**: remove the convexity assumption in the previous inequality
The trace operator

Let $\Omega$ be a Lipschitz domain and let us consider the norm of the trace operator $\tau : H^1(\Omega) \longrightarrow L^2(\partial \Omega)$. Computation of its norm leads to consider the eigenvalue

$$\frac{1}{\|\tau\|} = \lambda(\Omega) := \min \left\{ \frac{\int_{\Omega} |\nabla u|^2 + u^2 \, dx}{\int_{\partial \Omega} u^2 \, d\sigma} ; \ u \in H^1(\Omega) \right\}$$

which corresponds to the eigenvalue problem of Steklov type

$$\begin{cases} -\Delta u + u = 0 & \text{in } \Omega \\ \frac{\partial u}{\partial n} = \lambda u & \text{on } \partial \Omega \end{cases}$$

Open problem: Prove that the ball maximizes $\lambda(\Omega)$ among sets of given volume. Known: (J. Rossi 2008) the ball is a critical point.
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