An isoperimetric inequality of Saint-Venant-type for a wedge-like membrane

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Isoperimetric Inequalities for a Wedge-Like Membrane, Annales Henri Poincaré, March 2013

Contents

1. Four Classical Inequalities: (i) (Faber-Krahn) for the fundamental mode of vibration of a membrane, λ ,(ii) (de Saint Venant) for "torsional ridigity", P,(iii) Pólya-Szegő (relating $\lambda, P, |D|$), (iv) Pólya-Szegő Conjecture (relating λ, P, proved by Kohler-Jobin)
2. Payne-Weinberger's Improvement on Faber-Krahn, and Payne's Interpretation in "Fractional Weinstein Space"
3. Pólya-Szegő Conjecture (Kohler-Jobin inequality) is stronger than Faber-Krahn.

4. What we should retain from this presentation: (a) We propose in this work a complete program for wedge-like membranes; (b) For convex cones in higher-dimensions. (Only part (a), in this presentation)

5. Problem is a model for "manifolds with density"

6. What we describe is a model for eigenvalue problems associated with degenerate elliptic operators.

7. Renewed interest in "wedge-like membrane" isoperimetric problems (Ratzkin, Treibergs, Brock, Chiacchio, Mercaldo, etc.) with strong connections to weighted isoperimetric inequalities.

What to retain:

Old:

$$\text{Kohler-Jobin} + \text{Saint-Venant} \implies \text{Faber-Krahn.}$$

New:

weighted Kohler-Jobin +weighted Saint-Venant $\implies$ Payne-Weinberger

Why care?

Consider right isoceles triangle with equal sides of unit length (Payne-Weinberger)

$$\alpha = 1, \lambda_1 \geq 45.0734$$

$$\alpha = 2, \lambda_1 \geq 47.6325$$

$$\alpha = 4, \lambda_1 \geq 45.9094$$

Faber-Krahn (among all domains): $\lambda_1 \geq 36.3368$

Faber-Krahn (among all triangles): $\lambda_1 \geq \frac{4\pi^2}{\sqrt{3}\,A} \approx 45.5858$

Exact value: $\lambda_1 = 49.350625$

Rayleigh-Faber-Krahn (1897, 1920, 1923) inequality Let $D \subset \mathbb{R}^d$. Consider,

$$\begin{aligned} \Delta u + \lambda u &= 0 \text{ in } D \\ u &= 0 \text{ on } \partial D. \end{aligned} \tag{1}$$

Rayleigh-Ritz Principle:

$$\lambda = \inf_{\phi \in W_0^1(D)} \frac{\int_D |\nabla \phi|^2 \, dx}{\int_D \phi^2 \, dx} \tag{2}$$

$$\lambda(D) \geq \lambda(D^*) = \frac{C_d^{2/d} \, j_{d/2-1,1}^2}{|D|^{2/d}} \tag{3}$$

where $j_{d/2-1,1}$ denotes the first positive zero of the Bessel function $J_{d/2-1}(x)$ and $C_d = \pi^{d/2}/\Gamma(1+d/2)$ the volume of the unit ball.

$$|D| = |D^*|$$

2. De Saint Venant Inequality: Consider,

$$\begin{aligned} -\Delta v &= 1 \text{ in } D \\ v &= 0 \text{ on } \partial D. \end{aligned} \qquad (4)$$

Torsional rigidity is defined by

$$P = \int_D v\, dx = \int_D \left|\nabla v\right|^2 dx. \qquad (5)$$

Rayleigh-Ritz Principle:

$$\frac{1}{P} = \inf_{\phi \in W_0^1(D)} \frac{\int_D \left|\nabla \phi\right|^2 dx}{\left(\int_D \phi\, dx\right)^2}. \qquad (6)$$

2. De Saint Venant Inequality (proved by Pólya in 1948, Cont'd)

$$P(D) \leq P(D^*) = \frac{|D|^{1+2/d}}{d\,(d+2)\,C_d^{2/d}}$$

For $d = 2$, $P \leq P^* = \frac{|D|^2}{8\pi}$.

3. Pólya-Szegő Conjecture, 1951 (proved by Kohler-Jobin, 1978)

$$P(D)\,\lambda(D)^{\frac{d+2}{2}} \geq P(D^*)\,\lambda(D^*)^{\frac{d+2}{2}} = C_d\,\frac{j^{d+2}_{\frac{d}{2}-1,1}}{d\,(d+2)}$$

For $d = 2$, the original conjecture:

$$P(D)\,\lambda^2 \geq \pi\, j_{0,1}^4/8 = \frac{16.7\pi}{4}$$

This was proved by Kohler-Jobin (1975, 1978).
For $d = 2$, There were improvements by Payne, Payne-Weinberger, Payne-Rayner (1972),

$$P(D)\,\lambda^2 \geq \frac{16\pi}{4}$$

4. Pólya-Szegő (1951)
From (4), and the Rayleigh-Ritz principle for $\lambda(D)$, it is clear that

$$P = \int_D v\,dx = \int_D \left|\nabla v\right|^2 dx = \frac{\left(\int_D v\,dx\right)^2}{\int_D \left|\nabla v\right|^2 dx} \leq |D|\,\frac{\int_D v^2\,dx}{\int_D \left|\nabla v\right|^2 dx} < \frac{|D|}{\lambda(D)}$$

Therefore

$$P(D)\,\lambda(D) < |D|$$

Combining Kohler-Jobin & Saint Venant improves Faber-Krahn.

Combining the Kohler-Jobin Theorem, and the St. Venant Inequality, it is clear one fairs better than Faber-Krahn, viz.

$$\begin{aligned} \lambda(D) &\geq \left(\frac{C_d}{P(D)\, d\,(d+2)}\right)^{\frac{2}{d+2}} j^2_{\frac{d}{2}-1,1} \\ &\geq \frac{C_d^{2/d}\, j^2_{d/2-1,1}}{|D|^{2/d}} \end{aligned}$$

Payne-Rayner inequality (1973): For $D \subset \mathbb{R}^2$

$$\frac{\|u\|_2^2}{\|u\|_1^2} \leq \frac{\lambda}{4\pi} \tag{7}$$

with equality for the disk. This is a reverse Hölder inequality.
Kohler-Jobin (1977, 1981): For $D \subset \mathbb{R}^d$

$$\frac{\|u\|_2^2}{\|u\|_1^2} \leq \frac{\lambda^{d/2}}{2d\, C_d\, j_{d/2-1,1}^{d-2}}, \tag{8}$$

with equality if D is a ball.
Chiti (1982) gave the most general version of this reverse Hölder inequality for $\|u\|_q / \|u\|_p, q \geq p > 0$.

Start with (6), applied to the fundamental eigenfunction u to obtain

$$\frac{1}{P} \leq \frac{\int_D \left|\nabla u\right|^2 dx}{\left(\int_D \ u\, dx\right)^2} = \frac{\int_D \left|\nabla u\right|^2 dx}{\int_D \ u^2\, dx} \frac{\int_D u^2\, dx}{\left(\int_D \ u\, dx\right)^2}.$$

Therefore, applying Chiti with $p = 1, q = 2$, we obtain

$$\frac{1}{P} \leq \frac{\lambda^{1+\frac{d}{2}}}{2dC_d j_{\frac{d}{2}-1,1}^{d-2}}$$

Or,

$$P\, \lambda^{\frac{d+2}{2}} \geq C_d\, \frac{j_{\frac{d}{2}-1,1}^{d+2}}{d\,(d+2)} > 2d\, C_d\, j_{\frac{d}{2}-1,1}^{d-2}$$

$$\sum_{m=1}^{\infty} \frac{1}{j_{\nu,m}^4} = \frac{1}{2^4(\nu+1)^2(\nu+2)}$$

Apply for $\nu = \frac{d}{2} - 1$. (See "The Rayleigh Function", Kishore, 1963, or the Lehmer Tables, 1943)

Wedge: $\mathcal{W}_\alpha = \{(r,\theta) : 0 \leq \theta \leq \pi/\alpha\}$, $\alpha \geq 1$

Payne-Weinberger inequality for wedge-like membranes (1955): Let $D \subset \mathcal{W}_\alpha$. Then

$$\lambda \geq \lambda^* = \Big(\frac{4\alpha(\alpha+1)}{\pi}\int_D h^2(r,\theta) r\,dr\,d\theta\Big)^{\frac{-1}{\alpha+1}} j_{\alpha,1}^2 \qquad (9)$$

where $h = r^\alpha \sin\alpha\theta$. Here (r,θ) are polar coordinates taken at the apex of the wedge, and $j_{\alpha,1}$ the first zero of the Bessel function $J_\alpha(x)$. Equality holds if and only if D is a circular sector $\mathcal{W}_\alpha$.

$$\lambda(D)|D|_h^{\frac{1}{\alpha+1}} \geq \lambda(D^*)|D^*|_h^{\frac{1}{\alpha+1}} \qquad (10)$$

where D^* denotes any circular sector. Here

$$|D|_h = \int_D h^2(r,\theta) r\,dr\,d\theta.$$

This inequality improves on the Faber-Krahn inequality for certain domains (as is the case of certain triangles) and has the interpretation of being a version of Faber-Krahn in dimension $2\alpha + 2$ for axially symmetric domains (Bandle, Payne have the details).
The proof of this inequality relies on a geometric isoperimetric inequality for the quantity

$$|D|_h = A_\alpha = \int_D h^2(r,\theta) r \, dr \, d\theta$$

which is optimized for the circular sector in $\mathcal{W}_\alpha$, and a carefully crafted symmetrization argument (weighted symmetric decreasing rearrangement).

Geometric inequality for wedge-like membranes

For $D \subset \mathcal{W}_\alpha = \{(r,\theta) : 0 \leq \theta \leq \pi/\alpha\}$, $\alpha \geq 1$.

$$\left(\frac{2\alpha}{\pi} \oint_{\partial D} h^2(r,\theta) ds\right)^{(2\alpha+2)/(2\alpha+1)} \geq \frac{4\alpha(\alpha+1)}{\pi} A_\alpha$$

with equality for the circular sector.

Obviously, one would like to see if one can improve the other three inequalities (of Saint Venant, Pólya-Szegő, Kohler-Jobin) for wedge-like membranes.

Our work: Three Problems for wedge-like membranes

For $D \subset \mathcal{W}_\alpha$, we consider

$$\mathcal{P}_1 : \begin{cases} \Delta u + \lambda u &= 0 \quad \text{in } D \\ u &= 0 \quad \text{on } \partial D, \end{cases}$$

$$\mathcal{P}_2 : \begin{cases} -div(h^k \nabla w) &= h^k f \quad \text{in } D \\ w &= 0 \quad \text{in } \partial D \cap \mathcal{W}, \end{cases}$$

Here $k > 0$ and $h(r,\theta) = r^\alpha \sin \alpha\theta$, as above, where the function f belongs to the weighted Lebesgue space $L^2(D, d\mu)$, and $d\mu$ is the measure defined by

$$d\mu = h^k(r,\theta)\, rdrd\theta = r^{\alpha k+1} (\sin \alpha\theta)^k \, drd\theta. \tag{11}$$

The case $k = 2; f \equiv 1$ of $\mathcal{P}_2$

$$\mathcal{P}_3 : \begin{cases} -div(h^2 \nabla w) &= h^2 \quad \text{in } D \\ w &= 0 \quad \text{in } \partial D \cap \mathcal{W}, \end{cases}$$

Three problems, cont'd

We claim that $\mathcal{P}_3$ is equivalent to

$$\mathcal{P}_4 : \begin{cases} -\Delta v &= h(r,\theta) \quad \text{in } D \\ v &= 0 \quad \text{in } \partial D \cap \mathcal{W}, \end{cases}$$

To see this, let $v = hw$, in $\mathcal{P}_3$.
Relative torsional rigidity is defined via the variational formulation

$$\frac{1}{P_\alpha} = \inf_{\phi \in W_0^{1,2}(D)} \frac{\int_D |\nabla \phi|^2 \, r dr d\theta}{\left(\int_D \phi h \, r dr d\theta\right)^2}. \tag{12}$$

which is in fact equivalent to

$$\frac{1}{P_\alpha} = \inf_{\phi \in W_2(D, d\mu)} \frac{\int_D |\nabla \phi|^2 \, d\mu}{\left(\int_D \phi \, d\mu\right)^2}, \tag{13}$$

where $d\mu = h^2(r,\theta) \, r dr d\theta$.

- Case $\alpha = 1$. In this case, D is such that $y > 0$, and $\mathcal{P}_4$ reduces to

$$\mathcal{P}_4 : \begin{cases} \Delta v + y = 0 & \text{in } D \\ v = 0 & \text{in } \partial D \cap \mathcal{W}, \end{cases}$$

With $v = y\,w$, the problem is then

$$\mathcal{P}_4 : \begin{cases} \Delta w + \dfrac{2}{y}\dfrac{\partial w}{\partial y} = -1 & \text{in } D \\ w = 0 & \text{in } \partial D \cap \{y > 0\}, \end{cases}$$

Let the function $\Phi(x_1, x_2, x_3, x_4)$ be defined by

$$\Phi(x_1, x_2, x_3, x_4) = w(x, y) \text{ where } \quad x = x_4; \quad y = \sqrt{x_1^2 + x_2^2 + x_3^2}.$$

This function has axial symmetry with respect to the x_4-axis. It is defined on

$$D_4 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 | x = x_4, y = \sqrt{x_1^2 + x_2^2 + x_3^2}, (x, y) \in D\}$$

Payne Interpretation in Fractional Weinstein Space, Cont'd

$D_4 \subset \mathbb{R}^4$ is obtained from D via rotation around the x-axis. The function Φ satisfies

$$\Delta_4 \Phi = -1 \text{ in } D_4, \quad \Phi = 0 \text{ on } \partial D_4.$$

Note that $d = 2\alpha + 2 = 4$. Let $dV = dx_1 dx_2 dx_3 dx_4$, and

$$P = \int_{D_4} \Phi dV$$

Then

$$P_1 = \int_D v\, y\, dxdy = \int_D w\, y^2\, dxdy = \frac{1}{4\pi} \int_{D_4} \Phi dV = \frac{P}{4\pi}$$

Payne Interpretation in Fractional Weinstein Space, Cont'd

Therefore, applying the previous inequalities for P

- Pólya-Szegő:

$$P < |D_4|\lambda^{-1}$$

So

$$\begin{aligned} P_1 &< \frac{1}{4\pi}|D_4|\lambda^{-1} \\ &= \frac{1}{4\pi}(4\pi)\left(\int_D x^2 dxdy\right)\lambda^{-1} \\ &= A_1\,\lambda^{-1} \end{aligned}$$

where $A_1 = \int_D y^2 dxdy$.

- Payne-Rayner:

$$P\lambda^3 \geq 8\frac{\pi^2}{2}j_{1,1}^2.$$

Therefore

$$P_1\lambda^3 \geq \pi j_{1,1}^2$$

Payne Interpretation in Fractional Weinstein Space, Cont'd

- Saint Venant:

$$P \leq \frac{\sqrt{2}|D_4|^{3/2}}{24\pi}$$

So

$$P_1 \leq \frac{1}{3}\left(\frac{1}{8\pi}\right)^{\frac{1}{2}} A_1^{3/2}.$$

The original interpretation in the case of λ was observed by Payne.

$$\lambda \geq \frac{1}{2}\left(\frac{\pi}{2A_1}\right)^{1/2} j_{1,1}^2$$

and is also optimized for the half-disk.

Payne Interpretation in Fractional Weinstein Space, Cont'd

- Case $\alpha = 2$. In this case, D is such that $x > 0, y > 0$, and $\mathcal{P}_4$ reduces to

$$\mathcal{P}_4 : \begin{cases} \Delta v + 2xy = 0 & \text{in } D \\ v = 0 & \text{in } \partial D \cap \mathcal{W}, \end{cases}$$

With $v = 2x\,y\,w$, the problem is then

$$\mathcal{P}_4 : \begin{cases} \Delta w + \frac{2}{x}\frac{\partial w}{\partial x} + \frac{2}{y}\frac{\partial w}{\partial y} = -1 & \text{in } D \\ w = 0 & \text{in } \partial D \cap \{x > 0, y > 0\}, \end{cases}$$

Let the function $\Phi(x_1, x_2, x_3, y_1, y_2, y_3)$ be defined by

$$\Phi(x_1, x_2, x_3, y_1, y_2, y_3) = w(x, y)$$

with $x = \sqrt{x_1^2 + x_2^2 + x_3^2}; \quad y = \sqrt{y_1^2 + y_2^2 + y_3^2}$. This function has x and y as axes of symmetry. It is defined on

$$D_6 = \{(x_1, x_2, x_3, y_1, y_2, y_3) \in \mathbb{R}^6 | x = \sqrt{x_1^2 + x_2^2 + x_3^2}, y = \sqrt{y_1^2 + y_2^2 + y_3^2}, (x, y) \in D\}$$

obtained via two "rotations" of D around the coordinate axes.

The function Φ satisfies

$$\Delta_6 \Phi = -1 \text{ in } D_6, \quad \Phi = 0 \text{ on } \partial D_6.$$

Note that $d = 2\alpha + 2 = 6$. Let $dV = dx_1 dx_2 dx_3 dy_1 dy_2 dy_3$, and

$$P = \int_{D_6} \Phi dV$$

Then

$$P_2 = 2\int_D v\, x\, y\, dxdy = 4\int_D w\, x^2\, y^2 dxdy = \frac{4}{(4\pi)^2}\int_{D_6} \Phi dV = \frac{P}{4\pi^2}.$$

Also

$$|D_6| = \int_{D_6} dV = (4\pi)^2 \int_D x^2 y^2\, dxdy = 4\pi^2 A_2$$

where $A_2 = 4\int_D x^2\, y^2\, dxdy$.

Payne Interpretation in Fractional Weinstein Space, Cont'd

Therefore, applying the previous inequalities for P

- Pólya-Szegő:

$$P < |D_6|\lambda^{-1}$$

  which leads to

$$P_2 < A_2\,\lambda^{-1}.$$

- Payne-Rayner:

$$P\lambda^4 \geq 12\frac{\pi^3}{6}\,j_{2,1}^4.$$

Therefore

$$P_2\lambda^4 \geq \frac{\pi}{2}j_{2,1}^4$$

Payne Interpretation in Fractional Weinstein Space, Cont'd

- Saint Venant:

$$P \leq \frac{6^{1/3}|D_6|^{4/3}}{48\pi}$$

which simplifies as

$$P_2 \leq \frac{1}{4}\left(\frac{1}{72\pi}\right)^{\frac{1}{3}} A_2^{4/3}.$$

Again the original interpretation in the case of λ was observed by Payne

$$\lambda \geq \frac{1}{2}\left(\frac{\pi}{12A_2}\right)^{1/3} j_{2,1}^2,$$

and isoperimetry holds for the last two inequalities for the quarter disk with the same A_2 as D.

Pólya-Szegő for wedge-like membrane

Theorem. For a wedge-like membrane $D \subset \mathcal{W}_\alpha$

$$P_\alpha \le |D|_h \, \lambda^{-1}$$

where

$$|D|_h = A_\alpha = \int_D h^2 dxdy$$

Proof. Start with

$$\mathcal{P}_4 : \begin{cases} -\Delta v &= h(r,\theta) & \text{in } D \\ v &= 0 & \text{in } \partial D \cap \mathcal{W}, \end{cases}$$

$$P_\alpha = \frac{\left(\int_D v\, h\, dxdy\right)^2}{\int_D |\nabla v|^2} \le \frac{\int_D v^2 \int_D h^2}{\int_D |\nabla v|^2}$$

$$\le \lambda^{-1} |D|_h$$

Another Proof inspired by Pólya-Szegő

Since the eigenfunctions $\{u_n\}_{n=1}^{\infty}$ form an orthonormal basis of $L^2(D)$, corresponding to the eigenvalues $0 < \lambda \equiv \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_n \to \infty$, one can write

$$|D|_h = \int_D h^2 \, dA = \sum_{n=1}^{\infty} \left(\int_D h \, u_n \, dA \right)^2 , \qquad (14)$$

and

$$P_\alpha = \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \left(\int_D h \, u_n \, dA \right)^2 . \qquad (15)$$

The result is then immediate from the ordering of the eigenvalues, viz.

$$P_\alpha < \frac{1}{\lambda_1} \sum_{n=1}^{\infty} \left(\int_D h \, u_n \, dA \right)^2 = \frac{1}{\lambda_1} \, |D|_h .$$

Key: Expand $h = \sum_{n=1}^{\infty} \alpha_n \, u_n$ with $\alpha_n = \int_D h u_n \, dA$, and $v = \sum_{n=1}^{\infty} \beta_n \, u_n$, then use Plancherel-Parseval.

Payne-Rayner for wedge-like membrane

Theorem. For a wedge-like membrane $D \subset \mathcal{W}_\alpha$

$$P_\alpha \lambda^{\alpha+2} \geq \frac{\pi}{\alpha} j_\alpha^{2\alpha}$$

Proof. Start with Rayleigh-Ritz

$$\frac{1}{P_\alpha} \leq \frac{\int_D |\nabla u|^2 \, r dr d\theta}{\left(\int_D uh \, r dr d\theta\right)^2}.$$

u being the fundamental eigenfunction. Then

$$\frac{1}{P_\alpha} \leq \frac{\int_D |\nabla u|^2 \, r dr d\theta}{\int_D u^2 \, r dr d\theta} \frac{\int_D u^2 \, r dr d\theta}{\left(\int_D uh \, r dr d\theta\right)^2}.$$

We need a reverse Hölder inequality.

Chiti for wedge-like membranes

Theorem. For a wedge-like membrane $D \subset \mathcal{W}_\alpha$

$$\left(\int_D u^q h^{2-q} dA\right)^{\frac{1}{q}} \leq K(p,q,\lambda,\alpha) \left(\int_D u^p\, h^{2-p} dA\right)^{\frac{1}{p}} \qquad (16)$$

with $K(p,q,2\alpha+2)$ as given in the Chiti statement i.e.

$$K(p,q,\lambda,\alpha) = \left(\frac{\pi}{2\alpha}\right)^{\frac{p-q}{pq}} \lambda^{(\alpha+1)\frac{q-p}{pq}} \frac{\left(\int_0^{j_{\alpha,1}} r^{(2-q)\alpha+1} J_\alpha^q(r) dr\right)^{\frac{1}{q}}}{\left(\int_0^{j_{\alpha,1}} r^{(2-p)\alpha+1} J_\alpha^p(r) dr\right)^{\frac{1}{p}}}.$$

Equality holds when D is the circular sector.

Proof of Payne-Rayner result

Apply this theorem with $p = 1, q = 2$ This takes the explicit form

$$\int_D u^2 dA \leq \frac{\alpha}{\pi j_{\alpha,1}^{2\alpha}} \lambda^{\alpha+2} \left(\int_D u\, h dA \right)^2 . \qquad (17)$$

Theorem (Hasnaoui-H.) For a wedge-like membrane $D \subset \mathcal{W}_\alpha$

$$P_\alpha \leq \frac{1}{\alpha+2}\left(\frac{\alpha\,|D|_h^{\alpha+2}}{4^\alpha(\alpha+1)^\alpha\,\pi}\right)^{1/(\alpha+1)}$$

Equality holds for the circular sector.
Scale-free version:

$$P_\alpha(D)|D|_h^{-\frac{2\alpha+4}{2\alpha+2}} \leq P_\alpha(D^*)|D^*|_h^{-\frac{2\alpha+4}{2\alpha+2}}. \tag{18}$$

This is a corollary (case $k=2$, $f=1$ of the following more general setting)

Dirichlet Boundary Value Problem for a wedge-like membrane

Consider again the more general

$$\mathcal{P}_2 : \begin{cases} -div(h^k \nabla u) & = & h^k f & \text{in } D \\ u & = & 0 & \text{in } \partial D \cap \mathcal{W}, \end{cases}$$

Here $k > 0$ and $h(r, \theta) = r^\alpha \sin \alpha\theta$, as above, where the function f belongs to the weighted Lebesgue space $L^2(D, d\mu)$, and $d\mu$ is the measure defined by

$$d\mu = h^k \, dA = r^{\alpha k+1} (\sin \alpha\theta)^k \, dr d\theta. \tag{19}$$

Let f be a smooth function defined in D, and $f^\star$ denote its weighted symmetrization. We let $\mu(D) = \int_D d\mu$, and S_0 be the sector such that $\mu(D) = \mu(S_0)$, with r_0 denoting the radius of S_0.

Theorem 1

Let u be the weak solution to problem $(\mathcal{P}_2)$ and v the function defined by

$$v(r,\theta) = v^\star(r) = \int_r^{r_0} \left(\int_0^\delta f^\star(\rho)\rho^{\alpha k+1}\, d\rho \right) \delta^{-(\alpha k+1)}\, d\delta, \quad (20)$$

which is the weak solution to the problem

$$\mathcal{P}_4 : \begin{cases} -div(h^k \nabla v) & = & h^k f^\star & \text{in } S_0 \\ v & = & 0 & \text{in } \partial S_0 \cap \mathcal{W}. \end{cases}$$

Then

$$u^\star(x,y) \leq v(x,y) \quad a.e \quad in \;\; S_0. \quad (21)$$

and

$$\int_D |\nabla u|^q d\mu \leq \int_{S_0} |\nabla v|^q d\mu, \qquad 0 < q \leq 2 \quad (22)$$

Prove of Saint Venant

Let r_0 as above, then $v_\star = w_\star \; h$ where

$$v_\star(r,\theta) = \frac{1}{4\alpha+4}(r_0^2 - r^2)\, h(r,\theta) \quad \forall (r,\theta) \in S_0. \qquad (23)$$

is the explicit solution of (the symmetrized problem on S_0)

$$\begin{cases} -\Delta v &= h \quad \text{in } S_0 \\ v &= 0 \quad \text{on } \partial S_0. \end{cases}$$

$$\begin{aligned} P_\alpha = \int_D vhdA &= \int_D wd\mu = \int_D |\nabla w|^2 d\mu \\ &\leq \int_{S_0} |\nabla w_*|^2 d\mu = \int_{S_0} v_* hdA = P_\alpha^*. \end{aligned}$$

Kohler-Jobin for wedge-like membrane (Hasnaoui-H., Preprint 2013)

We also have the isoperimetric result improving Payne-Weinberger

$$P_\alpha(D)\lambda^{\alpha+2}(D) \geq P_\alpha(D^*)\lambda^{\alpha+2}(D^*) = \frac{\pi}{16\alpha(\alpha+1)^2(\alpha+2)} j_{\alpha,1}^{2\alpha+4}. \quad (24)$$

Theorem 2

Let u be the solution of problem $\mathcal{P}_2$. Then

(1) For $p > 1 + \frac{\alpha k}{2}$,

$$\text{ess sup}\,|u(r,\theta)| \le \mu(D)^{\frac{2}{\alpha k+2}-\frac{1}{p}} \, \frac{p(\alpha k+2)}{C(\alpha,k)^2\,(2(p-1)-\alpha k)} \left(\int_D |f|^p d\mu\right)^{\frac{1}{p}}$$

(2) For $1 < p < \frac{2(\alpha k+2)}{\alpha k+4}$, and $q = \frac{p(\alpha k+2)}{\alpha k+2-p}$, one has

$$\int_D |\nabla u|^q d\mu \le \mathcal{A}\, C^{-q}(\alpha,k) \left(\int_D |f|^p d\mu\right)^{\frac{q}{p}},$$

where

$$\mathcal{A} = \frac{p}{q(p-1)} \left(\frac{\Gamma\left(\frac{pq}{q-p}\right)}{\Gamma\left(\frac{q}{q-p}\right)\,\Gamma\left(\frac{p(q-1)}{q-p}\right)}\right)^{\frac{q}{p}-1},$$

$$C(\alpha,k) = \left(\frac{(\alpha\,k+2)^{\alpha\,k+1}}{\alpha} B\left(\frac{1}{2},\frac{k+1}{2}\right)\right)^{1/(\alpha k+2)} \qquad (25)$$

and B denoting the Euler Beta function.

A Geometric Isoperimetric Inequality

Proposition. Let D be a bounded subset of $\mathcal{W}$ with a C^1-boundary. Then, for any nonnegative number p, we have

$$\int_{\partial D} h^k(r,\theta)\sqrt{dr^2 + r^2 d\theta^2} \geq C(\alpha,k)\left(\int_D h^k(r,\theta) r dr d\theta\right)^{(\alpha k+1)/(\alpha k+2)}.$$

With

$$C(\alpha,k) = \left(\frac{(\alpha\, k+2)^{\alpha\, k+1}}{\alpha} B\left(\frac{1}{2},\frac{k+1}{2}\right)\right)^{1/(\alpha k+2)}.$$

Equality holds if and only if D is a circular sector of angle $\frac{\pi}{\alpha}$.

Remark: $k=0$, $\alpha \geq 1$; see Bandle's book (α-symmetrization); Lions-Pacella $k=0$, higher d using Brun-Minkowski; Payne-Weinberger $k=2$, $\alpha \geq 1$; Maderna-Salsa $\alpha = 1$, $k \geq 0$