Interplay of convex geometry and Banach space theory

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1 Convex geometry and Banach space theory

There are traditionally many interactions between the convex geometry community and the Banach space community. In recent years, work is being done as well on problems that are related to notions and concepts from other fields. The interaction of convex geometry and Banach space theory, and also with other areas, is due to high dimensional phenomena which lie at the crossroad of convex geometry and Banach space theory. But even though there are close ties between the two communities, we felt that much more could be achieved for both areas by having a fresh look, together, at some of the main problems of the two areas. That was exactly the topic of the workshop.

Banach space theory consists essentially of two subfields, the local theory of Banach spaces and the infinite dimensional theory of Banach spaces. Of course, there are interactions between the two. Relevant for the conference was only the local theory of Banach spaces. There, one tries to find out to what extend the finite dimensional subspaces of a given Banach space determine the nature of the Banach space.

Of particular importance to decide on the last mentioned aspect is the role played by invariances. We give a simple example that explains this. The type p constant of a Banach space X [30] is the infimum over all constants C such that for all n and all $x_1, \dots, x_n \in X$

$$\int_0^1 \left\| \sum_{i=1}^n r_i(t) x_i \right\| dt \leq C \left(\sum_{i=1}^n \|x_i\|^p \right)^{\frac{1}{p}} .$$

The cotype q constants are defined in the same way, with the inverse inequality. These constants depend only on the finite dimensional subspaces of the given Banach space and they describe geometric properties of the space. It is very easy to compute these constants for L_p-spaces and one gets immediately: If infinite dimensional L_p and L_q-spaces are isomorphic (i.e. there is a continuous, linear isomorphism) then $p = q$. To prove this result without the invariants of type and cotype is quite complicated.

We give another example. The celebrated theorem of Dvoretzky states that every infinite dimensional Banach space has subspaces of all finite dimensions that are are almost Euclidean. For finite dimensional normed spaces this theorem has the following reformulation: In every finite-dimensional normed space there is a subspace that is up to some prescribed error Euclidean and its dimension is at least of the order of the logarithm of the dimension of the space. More formally, there is a constant $c > 0$ such that for all $n \in N$, for all n-dimensional, normed spaces X and all $\epsilon > 0$ there is a subspace E of X with

$$d(E, \ell_2^{dim(E)}) \leq 1 + \epsilon \qquad \text{and} \qquad \dim(E) \geq c\epsilon^2 \ln(n),$$

where $d(E, \ell_2^{dim(E)})$ denotes the Banach-Mazur distance between E and the Euclidean space $\ell_2^{dim(E)}$.

This can be improved provided that the cotype 2 constant of the considered space is small. Then the almost Euclidean subspaces are of the order of the dimension of the considered space. More formally, there is a constant $c > 0$ such that for all $n \in N$, all n-dimensional, normed spaces X and all $\epsilon > 0$ there is a subspace E of X with

$$d(E, \ell_2^{dim(E)}) \leq 1 + \epsilon \qquad \text{and} \qquad \dim(E) \geq c\epsilon^3 \frac{n}{(C_2(X))^2},$$

where $C_2(X)$ denotes the cotype 2 constant of X.

Since $C_2(\ell_1^n) = \sqrt{2}$ there are subspaces E of ℓ_1^n with $d(E, \ell_2^{dim(E)}) \leq 1 + \epsilon$ and $\dim(E) \geq c\epsilon^3 n$.

In general, geometric invariants depend in a specific way on the dimension. This dependence has usually a geometric interpretation. In the finite dimensional setting, the geometric properties (of finite dimensional) normed spaces, are determined by their unit balls. These unit balls are convex bodies, i.e. compact convex subsets of R^n with non-empty interior, that are 0-symmetric. As such they are objects of research in convex geometry.

Affine invariants are among the most important tools of geometric analysis. Initially, affine invariants were systematically studied within affine differential geometry. Then there followed a systematic study by methods from partial differential equations of Monge-Ampère type, initiated by Calabi, Pogorelov, Chen and Yau. More recently, due, to a large extend, to powerful affine isoperimetric inequalities, affine invariants started to play a central role in convex geometry and geometric analysis, and certain partial differential equations, both parabolic and elliptic, became inherently related to them. Consequently, there have been numerous applications of affine invariants in the asymptotic theory of convex bodies, differential geometry, ordinary and partial differential equations and even in seemingly unrelated fields like geometric tomography and image processing.

Some of the most powerful affine invariants, the L_p affine surface areas [27, 28, 31, 40, 42, 43], arise in the the L_p-Brunn-Minkowski theory, which was initiated by Lutwak and Fiery. Besides their intrinsic interest, they are also valuations, a theory with tremendous recent growth e.g., [1, 6, 19, 24, 25, 39]. These invariants are naturally and essentially related to affine isoperimetric inequalities.

Though there has been great progress of the basic problems in convex geometry and local Banach space theory, due often to insight gained from studying the right affine invariants, fundamental problems remain open. We list some of them: Petty's conjectured projection inequality, the reverse Blaschke-Santaló inequality, the reverse centroid body inequality. One of the most important problems in local Banach space theory and convex geometric analysis is to determine the isotropy constant [9]. This problem has many equivalent formulations, one of which is the slicing problem:

Is it true that for every 0-centrally symmetric, convex body K in R^n of volume 1 there exists a hyperplane H through 0 such that $K \cap H$ has an $n-1$-dimensional volume bigger than some universal constant?

It is crucial to know the order of magnitude of this constant. Not only do many other major open problems in asymptotic geometric analysis (like the "isomorphic Busemann-Petty" problem, the variance conjecture, the Cheeger constant of a log-concave measure or the determination of the M-position of a convex body) depend on the answer to this question. But also it is crucial to e.g. computer science as it is directly related to the famous Kannan-Lovasz-Simonovits conjecture. Since the slicing problem is related to many different questions in many areas, it also leads to new open problems and new affine invariants.

2 Recent Developments and Open Problems. Scientific progress made during the workshop

We will list in this section some of the important problems in convex geometry and Banach space theory and indicate, for each problem listed, how it was addressed during the workshop and, if so, what progress has been made.

Throughout, $\mathcal{K}_n$ denotes the set of all convex bodies in R^n (i.e., compact convex subsets of R^n with nonempty interior).

A fundamental question in convex geometry and Banach space theory is how the volume of a convex body is distributed. This is not only important to compute the volume of a convex body, but also to compute volumes of other convex bodies naturally associated with a given convex bodies. We will illustrate this point below. Since a convex body seems to have a very simple structure this questions appears to be odd. A closer look however reveals that this is indeed a very difficult question.

A first example centers around the volume product of a convex body and the Blaschke-Santaló inequality. The volume product for $K \in \mathcal{K}_n$

$$\inf_{z \in K} vol_n(K) vol_n(K^z)$$

is an affine invariant. Blaschke [7] proved for dimension 3 and Santaló [38] for higher dimensions that

$$\inf_{z \in K} vol_n(K) vol_n(K^z) \leq (vol_n(B_2^n))^2.$$

In 1939 Mahler conjectured that the minimum of the volume product when restricted to centrally symmetric convex bodies is attained for the cube and the crosspolytope. Bourgain and Milman [11] proved in 1986 a quantitative version of the Mahler conjecture. They showed that there is a constant $c > 0$ such that for all $n \in N$

$$\frac{c^n}{n^n} \leq \inf_{z \in K} vol_n(K) vol_n(K^z).$$

For certain classes of convex bodies Mahler's conjecture was verified. Saint-Raymond showed this for the class of unconditional convex bodies [37] and Reisner showed that in the class of zonoids the Mahler conjecture holds [35]. For a long time there was no further improvement. In 2008 Kuperberg [23] proved the inequality of Bourgain-Milman by methods from Differential Geometry. The constant that he obtained is rather good, though not optimal. In 2009 Nazarov also proved the inequality by Bourgain and Milman [34]. He used methods of classical and harmonic analysis.

Very recently, some qualitative results have been established. Stancu [41] showed that a convex body with C^2-boundary and everywhere strictly positive curvature cannot be a minimizer for the volume product. This result has been generalized by Reisner, Schütt and Werner [36]. They showed that a body that has at least one boundary point with strictly positive generalized curvature cannot be a minimizer for the volume product.

Aside form these just mentioned connections of the Blaschke Santaló and Mahler inequalities to differential geometry, harmonic analysis, further links have been found. Indeed, just before the conference Artstein finished a joint paper with with Roman Karasev and Yaron Ostrover [2] in which they established a relationship between the Mahler conjecture and a problem in simplectic geometry. She spoke about their results at the workshop. In fact, Artstein together with her coauthors, showed that the Mahler conjecture and a symplectic isoperimetric conjecture are related. They linked symplectic and convex geometry by relating two seemingly different open conjectures: a symplectic isoperimetric-type inequality for convex domains, and Mahlers conjecture on the volume product of centrally symmetric convex bodies. More precisely, they showed that if for convex bodies of fixed volume in the classical phase space the Hofer-Zehnder capacity is maximized by the Euclidean ball, then a hypercube is a minimizer for the volume product among centrally symmetric convex bodies.

Further contributions on the topic “Blaschke Santaló and Mahler inequalities” were presented by Weberndorfer and Zvavitch.

An important question related to the distribution of the volume in a convex body is the already mentioned slicing problem, which is equivalent to the determination of the isotropy constant. We now give more details on this problem. In fact, we present here a more general framework. A probability measure μ on R^n is log-concave if for all compact sets A and B and all $\lambda \in (0,1)$

$$\mu((1-\lambda)A + \lambda B) \geq \mu(A)^{1-\lambda} \mu(B)^{\lambda}.$$

A log-concave measure is isotropic if for all $\theta \in S^{n-1}$

$$\int_{R^n} \langle \theta, x \rangle^2 d\mu(x) = 1.$$

In particular, for all convex bodies K in R^n of volume 1 the measure

$$\mu_K(A) = vol_n(K \cap A)$$

is log concave. Let f be the density of a log-concave measure μ. Then the isotropy constant of μ is

$$L_\mu = \left(\sup_{x \in R^n} f(x) \right)^{\frac{1}{n}}.$$

The isotropy constant of a convex body is defined accordingly. Bourgain [10] asked whether there is an absolute constant $c > 0$ such that for every $n \in N$ and every log-concave measure on R^n we have for the isotropy constant of μ

$$L_\mu \leq c \ ?$$

Bourgain proved that

$$L_K \leq c \sqrt[4]{n} \ln n.$$

Klartag was able to remove the logarithmic factor. During the workshop, there were also contributions towards this problem. For instance, Pivovarov studied in his talk projections of convex bodies instead of intersections with hyperplanes.

Another question about the distribution of the volume of a convex body is the Busemann-Petty problem: *Considering two convex bodies such that the intersection of every central hyperplane of the first body has a smaller measure than the one of the second body. Does the first body have smaller volume than the second?* This was answered through an effort of several researchers [16, 17, 44, 45]. In dimensions 1 to 4 the answer is positive, for higher dimensions the answer is negative.

Aside from having a surprising answer, the Busemann-Petty problem has sparked a lot of research. For instance, as the notion of intersection body was key to the solution of the Busemann-Petty problem, interest in those bodies was renewed. The intersection body of a convex body K in R^n is the body whose radial function for $\xi \in S^{n-1}$ is given by

$$vol_{n-1}(K \cap \{x \in R^n | \langle x, \xi \rangle = 0\}).$$

Lutwak [26] introduced intersection bodies, and showed that the Busemann-Petty problem has a positive solution in a given dimension if and only if every symmetric convex body is an intersection body. Koldobsky [22] showed that the unit balls of ℓ_p^n, $1 < p \leq \infty$, are intersection bodies for $n = 4$, but are not intersection bodies for $n \geq 5$. This observation turned out to be the key in the solution of the Busemann-Petty problem.

During the workshop, Koldobsky and Yaskin addressed problems concerning intersection bodies and Dann problems concerning the Busemann-Petty problem.

Another aspect of the distribution of the volume of a convex body is the covariogram problem. The covariogram g_k of a set K in R^n is the function that associates to each $x \in R^n$ the volume of the intersection $K \cap K + x$. The question is whether g_K determines K.

Bianchi spoke about this problem and reported on progress there.

A particular kind of invariants in convex geometry are valuations of convex bodies. In his 1900 ICM Address, David Hilbert asked in his Third Problem whether an elementary definition for volume of polytopes is possible. Max Dehn's solution in 1901 makes critical use of the notion of valuations, that is, of functions $\Phi : \mathcal{S} \to R$ that satisfy the inclusion-exclusion relation

$$\Phi(C \cup K) + \Phi(C \cap K) = \Phi(C) + \Phi(K),$$

whenever $K, C, K \cap C, K \cup C \in \mathcal{S}$ and where $\mathcal{S}$ is a collection of sets. Dehn solved Hilberts Third Problem by constructing a rigid motion invariant valuation which vanishes on lower dimensional sets and is not equal to volume (under any normalization). Since then investigations of valuations have been an active and prominent part of mathematics. A systematic study of valuations was initiated by Hadwiger, who was in particular interested in classifying valuations on the set $\mathcal{K}^n$ of convex bodies in R^n. Probably the most famous result on valuations is the Hadwiger characterization theorem.

Theorem [20]. *A functional $\Phi : \mathcal{K}^n \to R$ is a continuous and rigid motion invariant valuation if and only if there are constants $c_0, c_1, \ldots, c_n \in R$ such that for every $K \in \mathcal{K}^n$*

$$Z(K) = c_0 V_0(K) + \cdots + c_n V_n(K),$$

where $V_0(K), \dots, V_n(K)$ are the mixed volumes (2).

This has been generalized by Ludwig and Reitzner. They characterized those valuations that are upper semi continuous and invariant under rigid motions as well as linear nondegenerated maps [25]. This led to a characterization of the affine surface area of a convex body K

$$\int_{\partial K} \kappa^{\frac{1}{n+1}} d\mu_{\partial K},$$

where κ is the generalized Gauß curvature and $\mu_{\partial K}$ the surface measure on ∂K.

The concept of valuations has been extended to function spaces. A function Φ defined on a lattice $(L, \wedge, \vee)$ and taking values in an abelian semigroup is called a valuation if for all $f, g \in L$

$$\Phi(f \wedge g) + \Phi(f \vee g) = \Phi(f) + \Phi(g). \tag{1}$$

This is connected to the geometrization of analysis.

Valuations, in their different aspects, were present in many talks of the worksop, so for instance, in the talks of Bernig and Stancu. They also play a role in the next circle of problems.

Now we want to explain some aspects of the geometrization of analysis. Much research has been devoted to the question how notions of convex bodies can be adequately expressed for functions. There is a correspondence between convex bodies on the one side and log-concave, s-concave and quasi-concave functions on the other side. The first breakthrough was the analogy between the Brunn-Minkowski inequality and the Prékopa-Leindler (or Borell-Brascamp-Lieb) inequality [4, 13]. Let $0 < \lambda < 1$ and A and B be bounded, measurable sets in R^n such that $(1-\lambda)A + \lambda B$ is also measurable. Then the Brunn-Minkowski inequality is

$$vol_n(A)^{1-\lambda} vol_n(B)^{\lambda} \leq vol_n((1-\lambda)A + \lambda B).$$

Now we formulate the Prékopa-Leindler inequality. For all $0 < \lambda < 1$ and all non-negative, measurable functions $f, g, h : R^n \to R$ with

$$f(x)^{1-\lambda} g(y)^{\lambda} \leq h((1-\lambda)x - \lambda y)$$

we have

$$\left(\int_{R^n} f(x)dx\right)^{1-\lambda} \left(\int_{R^n} g(x)dx\right)^{\lambda} \leq \int_{R^n} h(x)dx.$$

Thus the inequality of Prékopa-Leindler is the functional form of the Brunn-Minkowski inequality.

For real numbers $t_1, \dots, t_m$ and convex bodies $K_1, \dots, K_m$ in R^n we define the Minkowski sum

$$t_1K_1 + \cdots + t_mK_m = \{t_1x_1 + \cdots + t_mx_m | x_1 \in K_1, \dots, x_m \in K_m\}.$$

The theorem of Minkowski states that the volume of a sum of convex bodies $K_1, \dots, K_m$

$$vol_n(t_1K_1 + \cdots + t_mK_m)$$

is a polynomial of degree n of the coefficients $t_1, \dots, t_m$. The coefficients $V(K_{i_1}, \dots, K_{i_n})$ of the polynomial

$$vol_n(t_1K_1 + \cdots + t_mK_m) = \sum_{i_1, \dots, i_n = 1}^{m} t_{i_1} \cdots t_{i_n} V(K_{i_1}, \dots, K_{i_n}) \tag{2}$$

are called mixed volumes of the convex bodies $K_1, \dots, K_m$. This equation has been extended to log-concave functions [8, 33]. In particular, for a log-concave function f, the Euclidean ball B_2^n in R^n and

$$W_i(f) = V(f, \dots, f, \chi_{B_2^n}, \dots, \chi_{B_2^n})$$

one obtains

$$W_0(f) = \int_{R^n} f dx \qquad W_1(f) = \int_{R^n} \|\nabla f\|_2 dx \qquad W_n(f) = \max_{R^n} f.$$

It turns out that W_i are valuations for log-concave functions (1).

While for some notions like the centroid this is clear, it is less obvious for the Blaschke-Santaló inequality or the affine isoperimetric inequality. Later functional forms of the Blaschke-Santaló inequality and its converse were studied by Ball, Artstein-Klartag-Milman, Fradelizi-Gordon-Reisner, Meyer, Lehec and Rotem. Various aspects of duality were studied initially by Ball and the by Artstein-Milman. More recently, area measure of log-concave functions and the first variation of volume have been studied by Klartag-Milman, Rotem and Fragalà and functional notions of mixed volumes and quermassintegrals by Milman-Rotem and Bobkov-Fragalà-Colesanti.

Milman and Rotem investigated s-concave functions and each of them reported on recent progress made in that subject. Colesanti spoke about functional notions of quermassintegrals and progress there.

For a Borel set $E \subseteq R^n$ and s with $0 < s < 1$ the fractional s-perimeter of E is given by

$$P_s(E) = \int_E \int_{E^c} \frac{1}{\|x-y\|_2^{n+s}} dxdy,$$

where $\| \ \|_2$ denotes the Euclidean norm on R^n. Bourgain, Brezis and Mironescu [12] and Dávila [14] have shown

$$\lim_{s \to 1} (1-s) P_s(E) = \alpha_n P(E),$$

where $P(E)$ is the perimeter of E and α_n a constant depending only on n. Monika Ludwig generalized this to arbitrary norms.

Another area of research deals with lattice points inside a convex body or points chosen randomly from a convex body. At the beginning of this research is the question whether it is possible to determine the volume of a convex body through an oracle. We choose randomly points in R^n and the oracle tells us whether the point is an element of the convex body. Is it possible to determine the volume of the convex body? Bárányi presented some research on the minimal perimeter of convex lattice polygons.

A key structural property of convex bodies is that of symmetry which is relevant in many problems. We only mention the still open Mahler conjecture about the minimal volume product of polar reciprocal convex bodies. The affine structure of convex bodies is closely related to the symmetry structure of the bodies. A systematic study of symmetry was initiated by Grünbaum in his seminal paper [18]. A crucial notion in his work is that of *affine invariant point*. A map $p : \mathcal{K}_n \to R^n$ is called an affine invariant point, if p is continuous and if for every nonsingular affine map $T : R^n \to R^n$ and all $K \in \mathcal{K}_n$

$$p(T(K)) = T(p(K)).$$

The centroid, the Santaló point and the center of the ellipsoid of maximal volume contained in a convex body are examples of affine invariant points. Affine invariant points allow to analyze the symmetry situation. In a nutshell: the more different affine invariant points, the fewer symmetries.

In his paper, Grünbaum asked several questions, some of them had been solved recently and others still remain open. Grünbaum asked whether the space of affine invariant points is infinite-dimensional. Intuitively, this is clear since there should be a lot of affine invariant points. Therefore, an important aspect in the proof is the construction of new affine invariant points. Actually, something stronger is true, namely that such convex bodies are actually dense in the metric space of convex bodies.

Grünbaum asked whether there are convex bodies such that their affine invariant points cover the whole space R^n. Clearly, the affine invariant points of a convex body are invariant under the symmetries of this body. Grünbaum asked whether the set of all those points, that are invariant under all symmetries of the convex body, represent its affine invariant points. More formally, Grünbaum conjectured [18] that for every $K \in \mathcal{K}_n$,

$$\mathcal{P}_n(K) = \mathcal{F}_n(K), \tag{3}$$

where

$$\mathcal{F}_n(K) = \{x \in R^n : Tx = x, \text{ for all affine } T \text{ with } TK = K\}.$$

In his talk Mathieu Meyer answered some of these questions [32].

3 Outcome of the Meeting

Already in the previous section, most of the scientific aspects of the outcome of the meeting were already addressed. We were particularly happy that almost all of the junior participants gave talks and want to want to emphasize again their contributions.

The PhD students that presented were Liran Rotem, Susanna Spektor and Manuel Weberndorfer. Liran Rotem talked about extensions of notions for convex bodies to the functional setting. Susanna Spektor gave a proof of a Khinchine inequality and Manuel Weberndorfer presented new inequalities for the asymmetric L_p volume product and the asymmetric L_p volume ratio.

Among the presenting postdocs was Susanna Dann. She gave a proof of the lower dimensional Busemann-Petty problem in the complex hyperbolic space. Ohad Giladi presented recent progress on Bourgain's quantitative version of a theorem by Ribe. And Joscha Prochno talked about combinatorial inequalities and subspaces of L_1.

In conclusion, there have been many interactions between the convex geometry community and the Banach space community in the last decade, but no systematic effort has been devoted to specifically address the particularly crucial notions of the various invariants occurring in the two fields. We concentrated on investigating invariants at the intersection of the two areas. In the past, researchers have worked on various problems making use of invariants of both fields and many new techniques have been developed in each of the areas. This workshop had a comprehensive look at some of the major problems of the mentioned areas and focus on a few important ones. We provided a forum for discussions with the objective of new collaborations and future research across the areas. In particular, as we invited a number of graduate students and postdocs, the workshop gave an opportunity for the young researchers to interact with people from related fields and form new connections.

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