Integrable Systems and Moduli Spaces

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August 25 - August 30, 2013

1 Overview of the Field

Moduli spaces in mathematics are used to parametrize families of geometric objects or structures. As soon as an object or structure admits a non-trivial deformation, it makes sense to talk about deformation parameters, or *moduli*. Moduli spaces naturally appear in all major branches of mathematics, especially in algebraic geometry, where they play a profound role. The string theory in modern theoretical physics, that is widely considered as a promising candidate to the theory unifying all fundamental interactions including gravity, also uses moduli spaces as one of the main tools.

On the other hand, the so-called "integrable" systems form a class of ordinary or partial differential equations which admit more efficient treatment than the others. The theory of such systems has been developed starting from 19th century, but got a real boost from the end of 1960's.

Theory of integrable system of (ordinary or partial) differential equations and algebraic geometry of moduli spaces of algebraic varieties co-existed separately and had very few common points for a long time. The situation changed drastically in the late 1970ies, when the methods of algebraic geometry were successfully applied to finding solutions of integrable systems. The construction of instantons, i.e. solutions to the self-dual Yang-Mills equation (Atiyah-Drinfel'd-Hitchin-Manin), explicit solutions to the KdV (Korteveg-deVries) and, more generally, KP (Kadomtsev-Petviashvili) equations (Novikov, Krichever, Dubrovin, Its, Matveev et al.) are the well-known examples of the close interaction established between the two theories.

In the mid-80ies it became clear that the theory of integrable systems can work surprisingly well in classical moduli problems. The classical Schottky problem (characterization of the Jacobian locus in the moduli space of principally polarized abelian varieties) was solved using the KP theory (Novikov, Shiota). Another example is an application of Yang-Mills and Yang-Mills-Higgs equations on algebraic curves to the geometry of the moduli space of vector bundles (Atiyah, Bott, Hitchin, Donaldson). This was a special case of a Hitchin system – the most universal class of finite-dimensional integrable systems introduced by Hitchin in 1987.

The real breakthrough took place in the beginning of 90ies, when Witten conjectured that the tautological intersection ring of the moduli space of (pointed) algebraic curves is governed by the KdV hierarchy. First proved by matrix integration (Kontsevich), and later verified by several other methods as well, Witten's conjecture still lacks a pure algebro-geometric proof!

Similar ideas were soon applied to another classical problem – namely, to the Hurwitz problem of enumeration of meromorphic functions on curves of a given genus, degree and topological type. It appeared that the generating function of the Hurwitz numbers satisfies another well-known integrable hierarchy called Toda equations (Okounkov, Pandharipande, Vakil et al.). A reduction of the Hurwitz problem to the intersection theory on the moduli space of algebraic curves was established by Ekedahl, Lando, Shapiro and Vainshtein.

Another famous integrability condition – the associativity equation of Witten-Dijkgraaf-Verlinde-Verlinde (WDVV equation) led to the new notions of moduli spaces of stable maps (Kontsevich), quantum cohomology and Frobenius manifolds (Dubrovin). This equation helped to solve a number of classical enumerative problems of algebraic geometry (Kontsevich et al.)

2 Recent Developments and Open Problems

Some of the recent applications of integrable systems to moduli problems:

- Development of the so-called "topological recursion" scheme by L.Chekhov, B.Eynard and N.Orantin. The topological recursion, starting from an arbitrary plane algebraic curve, leads to a hierarchy of "multi-point correlation functions", which turn out to encode (depending on the choice of the initial curve) a lot of important geometrical and combinatorial information Hurwitz numbers, Gromov-Witten and Donaldson-Thomas invariants etc.

- Development of the links between spaces of stability conditions of Calabi-Yau manifolds and triangulated categories by T.Bridgeland and I.Smith. In particular, spaces of quadratic differentials arise naturally in this framework.

- Witham equations were used to estimate the dimension of complete subvarieties and vanishing of certain tautological classes on the moduli space of algebraic curves (Grushevsky-Krichever). The main role in the Grushevsky-Krichever construction is played by real-normalized meromorphic differentials on Riemann surfaces

- Study of the isomondromic tau function of a class of Riemann-Hilbert problems associated to Hurwitz spaces allowed to obtain a non-trivial new relation in the Picard group of the space of admissible covers and the moduli space of holomorphic differentials on algebraic curves (Kokotov, Korotkin, Zograf). This result has lead G. van der Geer and A.Kouvidakis to solution of the problem of expressing the class of the so-called Mumford's M_3 divisor on the moduli space of Riemann surfaces of given genus.

3 Presentation Highlights

The conference talks can be tentatively divided into three groups. The talks from the first group (J.Hurtubise, E.Previato, D.Zakharov, A.Voronov, G. van der Geer, G.Farkas, R. de Jong, A.Schmitt, D.Zvonine) were mostly devoted to classical problems of algebraic geometry solved mostly by classical algebro-geometric methods. However, many constructions presented by speakers of this group were directly related to structures arising in integrable systems. In particular, real Riemann surfaces discussed by J.Hurtubise arise in the study of real algebro-geometric solutions of equations of KP type; the sigma-function and Abelian integrals discussed by E.Previato appear in her studies of algebraic solutions of dispersionless KP hierarchy. The talk by G. van der Geer, devoted to computation of Mumford's class M_3 on the moduli space of Riemann surfaces, was based on expression for the Hodge class on spaces of admissible covers in terms of boundary divisors, obtained by A.Kokotov, D.Korotkin and P.Zograf using an appropriate isomonodormic tau-function. Moduli spaces of Higgs bundles discussed by A.Schmitt are directly related to Hitchin's systems, which form the most general known class of integrable systems etc.

The second group of talks was devoted to the Chekhov-Eynard-Orantin topological recursion and related issues. This recursion and its relationship to various problems of geometry and combinatorics was, probablt, the most popular subject of the conference; its various aspects were discussed in talks by M.Mulase, P.Norbury, M.Kazarian, S.Shadrin, D.Zvonkine and G.Borot. In particular, M.Kazarian and S.Shadrin reported results of their joint work showing that the topological recursion equations are actually equivalent to an appropriate Virasoro consraint imposed on a partition function.

The third group of talks (by A.McIntyre, M.Moller, D.Chen, M.Basok, C.Norton) was devoted to application of analytical tools stemming from the theory of integrable and dynamical systems to the theory of moduli spaces.

The workshop was opened by J.Hurtubise who gave the survey of his recent joint work with I.Biswas, J.Huisman, O.Garcia-Prada and F.Schaffhauser, devoted to the theory of real vector bundles on real algebraic curves, with particular emphasis on topology of corresponding moduli spaces, and differences with the usual complex case.

In the next talk S.Grushevsky was presenting results of his recent joint work with I.Krichever. This work was inspired by the theory of Whitham equations aring in perturbation of classical integrable systems. The machinery of the theory of algebro-gemetric solutions of Whitham equations is based on meromorphic abelian differentials on Riemann surfaces, having all real periods. It turns out that by using these differentials one can prove highly non-trivial results about dimension of compact complex subvarieties of moduli spaces of Riemann surfaces (the Diaz theorem).

E.Previato presented her recent results on links between moduli spaces (in particular, the Gauss-Manin connection) and algebraic solutions of dispersionless KP hierarchy. In particular, Kleininan sigma-functions turn out to play an important role.

F.Soloviev presented recent results (joint with B.Khesin) lying at the intersection of two areas: the classical integrable systems theory and the theory of dynamical systems. He discussed a generalization of the famous two-dimensional pentogarm map to higher dimensions, and proved their integrability in the cases of closed and non-closed (so-called twisted) polygons. It turns out that in dimension d there exist d-1 natural generalizations of the pentogram map (the so-called “dented” pentagram maps). All of them, together with their continuous limits, turn out to be integrable, i.e. Lax representation with spectral parameter was found, which opens ways to complete description of such maps.

A.McIntyre described his recent joint results with J.Park. These results generalize classical link between Chern-Simons functional on compact 3d manifold, and the eta-invariant of the manifold to the case of Schottky manifolds (i.e. 3d manifolds whose conformal boundary is a compact Riemann surface). Namely, it tuns out that, under an appropriate choice of framing near the boundary, the Chern-Simons action can be expressed via determinant of laplacian at the boundary, Zograf’s F-function, and the Bergman tau-function on spaces of holomorphic differentials on Riemann surfaces (the holomorphic differential enters the scene since it is used to define framing near the conformal boundary). these results provide natural generalization of previous genus 1 results obtained by Yoshida.

The closing talk of the day was given by D.Zakharov, who discussed how to obtain tautological relations on moduli spaces of Riemann surfaces of compact type by pullback of relations on the universal Grassmanian via the Abel-Jacobi map.

During the second day the first two talks were devoted to various aspects of Chekhov-Eynard-Orantin topological recursion.

M.Mulase gave a few introductory examples of the topological recursion and appearance of “quantum” algebraic curves in its context.

In the next talk this topic was further developed by P.Norbury who studied the partition function of Gromov-Witten invariants of CP^1. Embedding of this problem in the scheme of topological recursion leads to another appearance of the “spectral curve”.

Then M.Möller spoke about Lyapunov spectrum of the ball quotients arising from cyclic coverings, which arise via construction of Deligne-Mostow. The sum of Lyapunov exponents was represented as ratio of certain interserction numbers and was computed by analysing the period map near boundary divisors. As a corollary, the classification of commesurability classes of all presently known non-arithmetic ball quotients was completed.

The talk ot T.Bridgeland was devoted to introduction in the spaces of stability conditions of triangulated categories. The main focus was put on the cases when the space of stability conditions coincides with a space of meromorphic quadratic differentials on Riemann surfaces.

In the talk by D.Chen was given on the theory of the Teichmüller flow. In this theory moduli spaces of Abelian and quadratic differentials appear naturally, and important dynamical characteristics of the Teichmüller flow (the Lyapunov exponents) can be expressed via integrals of certain natural classes over the moduli spaces. This makes study of connected components of the corresponding spaces important. In his talk, D.Chen proposed a new way to classify completely the connected components of moduli spaces of quadratic differentials in low genera.

V.Bouchard talked about generating function for the so-called orbifold Hurwitz numbers; these numbers equal to the numbers of inequivalent branched coverings which have all simple branch points, except one point which carries a special "r-orbifold" branching structure. The "mirror" conterpart of this generating function falls within the framework of Chekhov-Eynard-Orantin topological recursion with a specially chosen non-compact spectral curve. The same spectral curve turns out to appear also in Gromov-Witten theory on orbifolds.

The "young researchers" session on Tuesday consisted of two talks. T.Bothner spoke about asymptotics of partition function for the six-vertex model. S.Chowdhury discussed an alternative ways of deriving Goldman's bracket between monodromy matrices of flat connections on Riemann surfaces, using the technique of hamiltonian theory of integrable systems.

The Wednesday session started from the talk of S.Lando, who spoke about computation of characteristic classes for spaces of meromorphic functions on algebraic curves (Hurwitz spaces). Corresponding generating functions turn out to be solutions of integrable hierarchies.

The goal of the next talk, given by M.Kazarian, was to clarify theb geometrical meaning of the Chekhov-Eynard-Orantin topological recursion. It was shown that the partition functions satisfying the equations of the CEO recursion are in one-to-one correspondence with Lagrangian subspaces in the infinite dimensional symplectic vector space. In particular, equations of CEO recursion can be encoded in an equation involving one of Virasoro generators. Moreover, this construction gives a new proof of the ELSV formula and Bouchard-Marino conjecture.

The last talk on Wednesday was given by S.Shadrin, who spoke about relationship between the Chekhov-Eynard-Orantin topological recursion and Givental's formula from the theory of genus expansion of Frobenius manifolds. In particular, this correspondence allows to give an alternative proof of the ELSV formula for Hurwitz numbers in terms of intersection theory on moduli spaces of curves.

The Thursday session started from the talk by A.Voronov, who discussed the relationship between Batalin-Vilkovisky formalism of gauge theory and cohomology of moduli spaces. On the basis of this correspondence a new conjecture on cohomology of moduli spaces is given.

Then D.Zvonkine spoke about derivation of cohomological relations between tautological classes on Deligne-Mumford compactification of the moduli space of Riemann surfaces with punctures. The family of relations obtained in the talk contains all relations known to this day; derivation of this family of relations uses the Frobenius manifold structure of the A2 type singularity, and the 3-spin Witten's class.

Then G.Borot spoke about semi-classical expansion of solutions of systems of linear ODE, depending on a small parameter. he shown that to an arbitrary system of linear ODE one can associate an appropriate kernel, such that corresponding correlation functions satisfy the loop equations (or, equivalently, equations of topological recursion). This general scheme is discussed in the context of (p, q) minimal models.

The next 3 talks were of more algebro-geometric nature. G. van der Geer discussed the old problem of computation of the class of Mumford's divisor M_3 in the Deligne-Mumford compactification of the moduli space of Riemann surfaces. An essential part of this computation is based on expression for the Hodge class on spaces of admissible covers, which was derived by Kokotov, Korotkin and Zograf using the formalism of tau-functions originated in the theory of isomonodromic deformations of linear differential equations.

G.Farkas in his talk discussed a complete birational classification of the universal theta-divisor over the moduli space of curves of genus g, in particular explaining how the nature of the theta-divisor changes at genus 12.

R. de Jong discussed natural cohomology classes on the universal family of Riemann surfaces, induced by the canonical flat symplectic form on the universal Jacobian. These cohomology classes are compared with natural cohomology classes arising from Arakelov metric. Relationship between these two classes was studied by Kawazumi, and involves certain conformal invariant a_g. The behaviour of a_g near the boundary of Deligne-Mumford compactification of the moduli space, as well as the relationship to Faltings delta-invariant, were discussed.

During the young researchers' session M.Basok gave an analytical proof of G.Farkas's formula for an important divisor on spin moduli spaces in terms of standard generators of the Picard group of these spaces. This approach is based on the notion of the Bergman tau-function.

C.Norton then discussed the behavior of real normalized holomorphic and meromorphic differentials on Riemann surfaces under degeneration.

During the last day of the conference two talks were given. A.Schmitt spoke about motives of moduli spaces of Higgs bundles and holomorphic chains. The goal was to describe various ways of computation of Betti numbers of the character varieties of representations of centrally extended fundamental groups of Riemann surfaces. These character varieties are homeomorphic to moduli spaces of Higgs bundles of rank r and degree 1. For rank 2 and 3 the Betti numbers were computed by Hitchin and Gothen, respectively. In the talk it was shown how to compute the Betti numbers for $r = 4$ by combing the strategy of Hitchin with the approach of Harder and Narasimhan.

The last talk of the conference was given by J.Harnad, who started from introduction into Sato-Segal-Wilson theory of tau-functions, viewed as sections of determinant line bundles over infinite Grassmanians and flag manifolds, or, equivalently, as fermionic vacuum state matrix elements. The second part of the talk was devoted to various applications of this construction. The first set of applications consists of various matrix integrals, like the Brezin-Hikami model, or the Kontsevich intergal (which also appears as a generating function of Gromov-Witten invariants of a point). The second class of application included the Donaldson-Thomas invariants of of Calabi-Yau 3-folds. The third class of applications contained certain deformations of the latter tau-functions, which determine probability measures on the path space of random partitions. Such deformations define determinantal random point processes of random partitions, which are special cases of Schur processes, used as models for crystal melting and growth.

4 Scientific Progress Made

The talks and informal discussion during the workshop appeared to be very useful for stimulating further research of the participants. Just a few examples:

– Discussions between T.Bridgeland, S.Grushevsky and D.Korotkin led to a complete descriptions of the forgetful map fibers of the moduli space of abelian differentials in terms of the homological coordinates.

– Communication between G.Farkas and M.Basok resulted in new promising ideas how to analytically approach the problem of computing the classes of divisors in the moduli space of spin curves.

– The insights shared by M.Kazarian with P.Zograf developed into realization of deep connections between enumerative problems (including those for various moduli spaces) and completely integrable systems (like Kadomtsev-Petviashvili and 2-Toda hierarchies, etc.).

This list can be continued much further.

5 Outcome of the Meeting

The meeting was very successful with many highly relevant talks and scientific discussions. The participants from different areas of algebraic geometry and the theory integrable systems found many important points of common interest. Conditions provided by BIRS were really excellent and stimulating for research.

The main results presented during the workshop can be found in the following publications.

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