BIRS WORKSHOP 13W5091: GEOMETRIC AND TOPOLOGICAL GRAPH THEORY

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1. Overview

Since the beginning of graph theory, mathematicians have drawn graphs as points connected by curves. Although graphs can be studied without recourse to these diagrams, many of our most interesting problems relate the network properties of graphs with the geometric properties of their drawings. For example, the Four-Colour Theorem (which motivated a considerable amount of classical graph theory) bounds the chromatic number of graphs which can be drawn in the plane with no two edges crossing each other. More recently, the graph minors project and computational research have provided us new methods and tools while opening up whole new avenues of important theoretical and practical questions.

This workshop, a continuation of the highly successful one held seven years ago (06w5067), was organized around these broad themes of topological and geometric graph theory. The talks shed light on many different facets of the theme: topics included crossing numbers (Radoslav Fulek, Sergey Norin), graph colouring (Luke Postle), structure theorems (David Wood, Vida Dujmović, Chun-Hung Liu), and computational complexity (Sergio Cabello, Zdeněk Dvořák). Some of the problems presented had purely geometric origins (János Pach on the Erdős distinct distances problem), while others applied topological techniques to combinatorial problems (Penny Haxell on Ryser's Conjecture). Several of the talks inspired new research collaborations during and after the workshop, including Matthieu Plumattaz' talk on graphs with no 0 mod 3 cycles and Jacob Fox' talk on interval minors.

One of the goals of this workshop was to bring together researchers working on topological and geometric graph theory and on computational topology in an effort to spark new collaborations and research projects. A number of graduate students and postdocs specializing in these areas also attended to learn about the relevant methods, results, and open problems.

2. Participants

Archdeacon, Dan	University of Vermont
Berger, Eli	Haifa University
Bokal, Drago	University of Maribor
Cabello, Sergio	University of Ljubljana
Churchley, Ross	Simon Fraser University
Dujmović, Vida	University of Ottawa
Dvořák, Zdeněk	Charles University, Prague
Edwards, Katherine	Princeton University
Ellingham, Mark	Vanderbilt University
Fox, Jacob	MIT
Fulek, Radoslav	Charles University
Gauthier, Gregory	Princeton University
Goddyn, Luis	Simon Fraser University
Haxell, Penny	University of Waterloo
Kim, Ringi	Princeton University
Klimošová, Tereza	University of Warwick
Kostochka, Alexandr	U. Illinois Urbana-Champaign
Li, Zhentao	École Normale Supérieure de Lyon
Liu, Chun-Hung	Georgia Tech
Mohar, Bojan	Simon Fraser University
Noel, Jonathan	University of Oxford
Norin, Sergey	McGill University
Pach, János	EPFL & Renyi Institute
Plumettaz, Matthieu	Columbia University
Postle, Luke	Emory University
Reed, Bruce	McGill University
Rus, Jernej	IMFM & Cosylab
Salazar, Gelasio	U. Autonoma de San Luis Potosi
Scott, Alex	University of Oxford
Seymour, Paul	Princeton University
Širáň, Jozef	Open University
Tóth, Géza	Renyi Institute
Whalen, Peter	Georgia Tech
Wood, David	Monash University
Wu, Hehui	Simon Fraser University
Yancey, Matthew	Institute for Defense Analysis
Yepremyan, Liana	McGill University
Yuditsky, Lena	McGill University
Zhao, Yufei	MIT

3. PROGRAM

The first day started with "5-minute presentation" of all participants, giving opportunity to learn about each other's work and current research interests. The day ended with the Open Problem Session. The subject of the remaining presentations is listed below.

Eli Berger *Relation between the maximal degree of a graph and the topological connectivity of its independence complex* Tuesday

Sergio Cabello *Parameterized complexity of 1-planarity* Tuesday

We consider the problem of finding a 1-planar drawing for a general graph, where a 1-planar drawing is a drawing in which each edge participates in at most one crossing. Since this problem is known to be NP-hard we investigate the parameterized complexity of the problem with respect to the vertex cover number, tree-depth, and cyclomatic number. For these parameters we construct fixed-parameter tractable algorithms. However, the problem remains NP-complete for graphs of bounded bandwidth, pathwidth, or treewidth.

This is a joint work with Michael Bannister and David Eppstein.

Zdeněk Dvořák *Coloring graphs on surfaces* Wednesday

A graph H is *t-apex* if $H - X$ is planar for some set $X \subset V(H)$ of size t. For any fixed t-apex graph H, we give a polynomial-time algorithm to decide whether a $(t+3)$-connected H-minor-free graph is colorable from a given assignment of lists of size $t+4$. The connectivity requirement is the best possible in the sense that for every $t \geq 1$, there exists a t-apex graph H such that testing $(t+4)$-colorability of $(t+2)$-connected H-minor-free graphs is NP-complete. Similarly, the size of the lists cannot be decreased, since for every $t \geq 1$, testing $(t+3)$-list-colorability of $(t+3)$-connected K_{t+4}-minor-free graphs is NP-complete.

This is a joint work with Robin Thomas.

Mark Ellingham *Partial duality for graph embeddings* Thursday

In 2009 Chmutov introduced a partial duality operation, using only a subset of the edges, for graph embeddings. The topological consequences of this operation have not yet been intensively investigated. We give an introduction to this idea, and discuss some recent work, including bounds on the genus of partial duals, and when a partial dual is a bouquet (1-vertex embedding).

This is joint work with Kenta Noguchi and Xiaoya Zha.

Jacob Fox *Stanley-Wilf limits are typically exponential* Tuesday

Radoslav Fulek *Recent Progress on Hill's Conjecture* Tuesday

In 1958, Hill conjectured that the minimum number of crossings in a drawing of K_n is exactly

$$Z(n) = \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor.$$

Generalizing the result by Ábrego et al. for 2-page book drawings, we prove this conjecture for plane drawings in which edges are represented by x-monotone curves. In fact, our proof shows that the conjecture remains true for x-monotone drawings of K_n in which adjacent edges may cross an even number

of times, and instead of the crossing number we count the pairs of edges which cross an odd number of times.

This is a joint work with Martin Balko and Jan Kynčl.

Penny Haxell *Extremal hypergraphs for Ryser's conjecture* Tuesday

A *packing* (or *matching*) in a hypergraph H is a set of pairwise disjoint edges of H. A *cover* of H is a set C of vertices that meets all edges of H. A famous open problem known as Ryser's Conjecture states that any r-partite r-uniform hypergraph should have a cover of size at most $(r-1)\nu(H)$, where $\nu(H)$ denotes the size of a largest packing in H. This was proved by Aharoni in 2001 for the case $r = 3$. Here we show that if equality holds in this case then H belongs to a special class of hypergraphs we call "home base hypergraphs". To prove this we need to study the topological connectedness of the matching complex of bipartite graphs.

This is a joint work with L. Narins and T. Szabó.

Alexandr Kostochka *Planar 4-critical graphs with exactly 4 triangles* Wednesday

We describe the (infinite) family of 4-critical planar graphs with exactly 4 triangles. This answers a question of Erdős from 1990 and of Axenov from 1970s. The proof uses the recent solution of Gallai's Conjecture from 1963 on k-critical graphs with few edges by Yancey and the speaker.

This is joint with O. Borodin, Z. Dvořák, B. Lidicky and M. Yancey.

Zhentao Li *Intersection graphs of axis parallel rectangles and Scott's conjecture* Thursday

Chun-Hung Liu *Structure theorems and well-quasi-ordering* Thursday

Motivated by well-quasi-ordering problems, we prove structure theorems for excluding a fixed graph H as a weak immersion or a topological subgraph, improving upon earlier results of Grohe and Marx, Dvořák, and Wollan. For topological minors our ultimate goal is an old conjecture of Robertson that for every integer k graphs with no topological minor isomorphic to the graph obtained from a path of length k by doubling every edge are well-quasi-ordered by the topological minor relation. We are able to prove the conjecture for graphs of bounded tree-width and reduce the general problem to graphs that possess certain kind of tree-decomposition. We expect that our proof for graphs of bounded tree-width will generalize to graphs possessing said decomposition, but the details of that have not yet been worked out at the time of submission.

Bojan Mohar *Rooted $K_{2,4}$ minors* Tuesday

Let G be a graph with four distinguished vertices t_1, t_2, t_3, t_4 called *terminals*. By a *rooted $K_{2,4}$ minor* in G we refer to a collection of six pairwise disjoint connected subgraphs T_1, T_2, T_3, T_4 and S_1, S_2 of G such that $t_i \in V(T_i)$ for $1 \leq i \leq 4$ and each T_i is adjacent to S_1 and S_2. These and more general rooted minors appear naturally in graph minors theory.

In the talk, a result about existence of rooted $K_{2,4}$ minors in planar graphs and the corresponding structure theorem will be outlined. Although the proof is both long and complicated, the structure turns out to be quite accessible. The problem is first reduced to the 3-connected case, and then it is shown that there are no rooted $K_{2,4}$ minors if and only if the graph can be reduced to one of five specific structures. This result gives rise to a good characterization for existence of rooted $K_{2,4}$ minors in planar graphs because no graph possessing a structure in the list can have a rooted $K_{2,4}$ minor.

Although this result is of independent interest, the main motivation is in its application for the problem of 4-terminal wye-delta reducibility in planar graphs.

This is a joint work with Lino Demasi.

Sergey Norin *Turán's brickyard problem and flag algebras* Tuesday

János Pach *Incidence results on the plane and distinct distances* Thursday

Matthieu Plumettaz *Graphs with no 0 mod 3 cycles* Thursday

Luke Postle *3-coloring and 3-list-coloring graphs on surfaces* Wednesday

Bruce Reed *Length of a longest cycle in a grid* Tuesday

Alex Scott *Hypergraphs of bounded disjointness* Tuesday

A k-uniform hypergraph is said to be intersecting if no pair of edges is disjoint. The maximal size of an intersecting k-uniform hypergraph with a given groundset is given by the beautiful and well-known theorem of Erdős, Ko, and Rado.

A k-uniform hypergraph is s-almost intersecting if every edge is disjoint from exactly s other edges. Gerbner, Lemons, Palmer, Patkós and Szécsi made a conjecture on the maximal number of edges in such a hypergraph. We prove a strengthened version of this conjecture and determine the extremal graphs. We also give some related results and conjectures.

This is a joint work with Elizabeth Wilmer.

Paul Seymour *Variants of Woodall's conjecture* Thursday

Géza Tóth *The Erdos-Szekeres theorem for lines* Thursday

Peter Whalen *Odd $K_{3,3}$ subdivisions in bipartite graphs* Thursday

David Wood & Vida Dujmović *Tree-decompositions on surfaces with applications* Thursday

This talk introduces the notion of the breadth of a tree decomposition of a graph. It is proved that every H-minor-free graph has a tree decomposition of bounded breadth if and only if H is apex. Applications of this result for queue layouts, 3-dimensional graph drawing, and non-repetitive chromatic number are given.

4. Workshop collaborations and inspirations

Participants were asked to share some of their thoughts about collaborations initiated during this workshop.

Dan Archdeacon. Following from talks with Luis Goddyn and Drago Bokal we've started a research project on John Conway's "Thrackle Conjecture" regarding the maximum crossing number of a graph. The opportunity to be together was essential to this collaboration. We were also helped in this by discussions with János Pach.

I also enjoyed Sergey Norin's discussion about many different drawings that achieve the minimum crossing number of complete bipartite graphs. I discussed this with Gelesio Salazar and have started working on a related problem about representing such drawings of general graphs in the projective plane. I had helpful discussions with Luis Goddyn and Mark Ellingham about embedding graphs with Hamiltonian cycles as faces. Finally, I will be pursuing ideas from David Wood's and Vida Dujmović's talks on the relationship between linear volume layouts of graphs and the track/queue number of a graph.

Drago Bokal. With Radoslav Fulek, we investigated the gap of the crossing number additivity over cuts, with Jernej Rus, we devised a roadmap that will lead to an efficient algorithm for counting stable traces in graphs, and with Dan Archdeacon and Luis Goddyn we investigated approaches to formulating the thrackle existence problem as a stack of constraint satisfaction problems that may lead to a new algorithm for recognizing thrackle graphs. Besides the actual work, there were several interesting talks presenting deep, although sometimes counterintuitive new results.

Vida Dujmović. Workshops of this format are the most productive venues for learning and exchange of ideas. For example, it is via the talk of Zdeněk Dvořák that I learnt about a new, and as yet unpublished, result by the speaker and Robin Thomas. Pat Morin, David Wood and myself needed this result as one of the key components in our recent research. Without meetings such as this BIRS meeting, learning about such a result would likely be much delayed and cause us to spend many hours attempting to prove an already known result. This is just one small example of a positive impact of the meeting. I also learnt about exciting new techniques applicable to the field, including application of flag algebras to the crossing number problem; started new collaborations, with Sergio Cabello for example; and had insightful exchange of ideas with many participants after my talk — some of which are likely to lead to new discoveries.

Zdeněk Dvořák. I was surprised by the result of Luke Postle that the potential method can be used to improve the bounds on size of embedded 4-critical graphs of girth 5. We also briefly discussed the possiblity of using the method for fractional or circular chromatic number, which we hope to work more on together in the future. The Erdős-Pósa-like problem for cycles in planar graphs raised by Bruce Reed was also inspiring. Although we came up with counterexamples to most reasonable formulations, I think a restriction to 3-connected planar graphs still has potential to give interesting results.

Greg Gauthier. Paul Seymour and I presented the following conjectures by Kalai and Meshulam:

- If a graph has no induced cycles of length a multiple of 3, then the number of even independent sets minus the number of odd independent sets is at most one in absolute value.
- If a graph has no induced cycles of length a multiple of 3, then the graph has bounded chromatic number (and, as a stronger conjecture, is also 3-colorable).
- If a graph has all of its induced subgraphs satisfying number of even independent sets minus number of odd independent sets is at most 1 in absolute value, then the graph has bounded chromatic number.

These conjectures generated much discussion. Zhentao Li, Paul Seymour, and others, including myself, worked on adapting Matthew Plumettaz's results for graphs with no cycles of length a multiple of 3 to the dual problem of having no bonds with a multiple of 3 edges. We discussed how to prove that there is no simple, 3-edge-connected graph with no bonds having a multiple of 3 edges. Additionally, many of the presentations not only presented interesting problems to consider, they also provided possible approaches for proving the Kalai and Meshulam conjectures in whole or in part.

Mark Ellingham. I found the workshop generally useful. Some specific highlights were as follows:

- I presented a problem on nets of triangular polyhedra, due to David Richter, at the problems session. One evening several participants discussed this problem and we believe we have solved it. If correct, this will probably lead to a publication (in which BIRS will be appropriately acknowledged).
- Alex Scott presented a five-minute talk in which he discussed geometric reconstruction problems. I have worked on graph reconstruction problems in the past, and I was not aware of how much had been done on geometric reconstruction, so this has re-awakened my interest in reconstruction problems.
- Luis Goddyn presented a problem on hamilton cycle embeddings of hypercubes. I am very interested in hamilton cycle embeddings. Dan Archdeacon began working on an idea to prove that the embeddings discussed by Luis Goddyn exist, and I will be very pleased if he can show that they do.
- Penny Haxell explained to me the details of the use of topological connectivity in the proof of results on hypergraph packings and related results. I will now be more alert to situations where the notion of topological connectivity might be useful in my own work.
- I found out there was a connection between some of the work I am doing on partial duality and some ideas used in the work by David Wood and Vida Dujmović. They know some early references for the background work for what I am doing, so this is useful information for when I write up my results.

Radoslav Fulek. I had many mathematical discussions and I learnt about several nice open problems and results that I was not aware of before the workshop. In particular, I have become seriously interested in Drago Bokal's problem on the crossing number of a graph and new methods for bounding crossing number of a complete bipartite graph based on flag algebras presented by Sergey Norin. I was also happy to discuss with Drago Bokal possible improvements of the algorithm from my joint work with János Pach on Conway's Thrackle Conjecture.

Penny Haxell. The talk of Eli Berger was particularly interesting to me, as the topological connectedness of the independence complex of a graph is an important parameter for many of the problems I work on, yet it seems very difficult in general to estimate. So hearing the details about his work on it was useful. Also I caught up with him in conversation on the status of some other projects he has, which are relevant to my work.

I had interesting mathematical conversations with various other participants, for example Mark Ellingham (about generalised duals, topological connectedness, its homological version etc.), Sasha Kostochka (about edge colouring, and in particular a nice new approach he has which gives, among other things, a nice and easily teachable proof of Vizing's Theorem), Joszef Širáň (about embeddable 3-graphs), and many others.

Because this meeting brought together quite a diverse group, I met some researchers for the first time, and appreciated the chance to find out what they work on.

Tereza Klimošová. I was impressed by the progress, presented in the talk of Matthieu Plumettaz, that was made towards the conjecture that the difference between the number of independent sets of even and odd size is at most one in graphs without induced cycle of length 0 mod 3. I worked on the problem a while ago and it was very useful for me to see the most recent progress and that people are still interested in the problem.

Sasha Kostochka. I learned not only about excellent results, but whole new (for me) topics to study and use. One example is the talk of Jacob Fox, where I learned the topic of Turán-type problems for permutations, apart from his outstanding result solving (disproving) several old conjectures. I also started a new project with Hehui Wu at the conference.

Zhentao Li. In their presentation, David Wood and Vida Dujmović presented a new width parameter, breadthwidth, and use it to clarify and categorize. I feel their result will be very helpful for article I am writing that use locally bounded treewidth.

Discussion with Sergey Norin was fruitful for a project I am about to start on a linear programming based approach to automating discharging arguments (in planar and possibly other clases of graphs). He suggested another class of inequalities need to be considered: those generated by Cauchy-Schwartz type inequalities. Otherwise, it is possible that some true inequalities cannot be obtained by linear combinations of "trivial" inequalities and Euler's formula.

After my own presentation, attendants (including Jacob Fox, János Pach, Paul Seymour, and Sasha Kostochka) spoke to me about possible directions in which the results can now be expanded. One question asked was if graphs excluding all cycles of length at least l are χ-bounded then does this imply that graphs excluding a path of long cycles are also χ-bounded (each joined to the next cycle at a cut vertex). This is motivated by the fact that the number of potentially χ-bounded classes is now reduced. Another question raised is a conjecture on the linear number of edges in a k-quasi-planar graph (a graph that can be drawn in the plane so no k edges pairwise cross). Finally, another question about finding the best χ-binding function for the intersection graph of filled axis-parallel rectangles. I was told the proof techniques used to prove χ-boundedness (with a quadratic bound) are similar to the ones we used in the result I presented.

Chun-Hung Liu. During the workshop Zdeněk Dvořák mentioned that there is no known proof to show that independence number of a planar graph is at least $\frac{n}{4.5}$ without using four color theorem. He also conjectured that there exists $c > 0$ such that the fractional chromatic number of every triangle-free planar graph is at most $3 - c$. I discussed this question with Hehui Wu and Luke Postle. Also, Paul Seymour asked a question about whether the chromatic number of a graph obtained by the union of two chordal graphs on the same vertex set is bounded by the sum of the chromatic number of these two chordal graphs. This question seems interesting and I might try to think about it.

Jernej Rus. At the beginning of this workshop I mentioned two open problems that I am currently interested in: characterizing graphs which admit antiparallel d-stable traces, and finding an efficient algorithm for counting non-equivalent double traces in graphs. I got Luis Goddyn interested in the first problem, which could lead to a future collaboration, and I started to work on a conjecture that graph G admits antiparallel d-stable trace if and only if G admits a spanning tree T such that every connected component of $G - E(T)$ has an even number of edges or contains a vertex v of degree at least $2d + 2$.

As for the second problem, I started a new collaboration with Drago Bokal. We managed to discuss and scheme three different algorithms using linear programming, backtracking and branch and bound methods.

Alex Scott. The Banff meeting was a great stimulus to my research. David Wood and I discovered some interesting connections between our work related to intersecting set systems; this is very likely to lead to further interactions between us (and between our students). The work of David Wood and Vida Dujmović on nonrepetitive colourings also started me thinking along new lines.

Paul Seymour. In joint work with various others (in the 2nd floor lounge) we settled an open problem raised during the problem session, a conjectured characterization of the outplanar triangulations that arise as nets of a planar triangulation. This may be written up as a paper.

The talk of Matthieu Plumettaz stimulated some other research. Plumettaz was interested in the graphs with no cycle of length a multiple of three, and we looked at the dual problem—which graphs have no bond of size a multiple of three? Again a group of several of us solved this.

The talk by Jacob Fox introduced a new containment relation on matrices, which can viewed as a minor relation on bipartite graphs with both sets of vertices ordered. I was happy to notice that exactly the same relation is what is needed for a different problem I am currently working on, trying to understand

the minimal induced subgraphs that have large clique minors. Fox's suggestion seems to lead to a bunch of new problems that are not yet explored, and I plan to give them to my graduate students.

Jozef Širáň. I learned a lot by attending the talks. The most inspiring one from the perspective of my interest in highly symmetric maps was the talk given by Mark Ellingham about partial duality. I liaised with Dan Archdeacon and we started looking at partial dualities which would preserve symmetries of a map.

Géza Tóth. Jacob Fox's talk was extremely interesting. It is really surprising that elementary (but very tricky) methods work for this deep problem about matrices with forbidden submatrices. These results are strongly related to the theory of geometric and topological graphs. Radoslav Fulek's talk was also very interesting; he described a generalization of the methods and formula of Lovász et al. that connects crossing numbers and k-sets to monotone (and even more general) drawings.

I hope I can use the ideas, methods, and results from these talks for my research on the theory of topological graphs. One particular problem where I hope to make some progress: it was recently shown that in a complete simple topological graph of n vertices there are always at least n empty triangles, but the best upper bound is $2n-4$, and it is very likely that the truth is (close to) the upper bound.

Peter Whalen. Many of the talks were very helpful for my research: I've spent several hours now collaborating with a small group of people on some results related to cycles of length 0 mod 3. Coming into this conference, I was working in the area of flat embeddings in graphs. A discussion with Paul Seymour led me to an old theorem of his that will hopefully shed some light on the problems I was working on.

David Wood. I heard about the recent breakthrough structural characterisation of apex minor free graphs due to Zdeněk Dvořák and Robin Thomas. This result was incredibly helpful to my research on layered decompositions and separators with Vida Dujmović and Pat Morin. Indeed, their result was exactly what we needed to turn the main conjecture in our paper into a theorem. This interaction would not have happened without this workshop.

The talk of Alex Scott was very relevant to a project that I am currently working on with a PhD student in Melbourne regarding generalisations of the Erdős-Ko-Rado theorem. Discussions with Alex pointed to many gaps in the literature that we were unaware of. Moreover, the results that my PhD student and I recently obtained enthused and encouraged Alex. I hope this will lead to future collaborations between Alex and myself and my students.

Matt Yancey. I got new ideas on how to proceed with 3-coloring sparse graphs with girth 4 from Luke Postle's presentation on 3-coloring sparse graphs with girth 5. I was also interested in a new problem presented by Sergio Cabello.

Yufei Zhao. My background is in extremal and probabilistic combinatorics and graph theory, and I was particularly excited to see so many talks at the intersection of these areas. The first talk of the workshop was by Jacob Fox, who proved very recently a rather surprising result disproving a long standing conjecture regarding permutation avoidance. Fox explained his approach in terms of interval-minors in matrices, in analogy with graph minors. Fox and I spent much of the remainder of the workshop discussing about extending these ideas further to refine the solution. Another talk that I enjoyed was by János Pach, who gave us an update on the Erdős distance problem, and I became aware of the many open problems that still remain in this area.

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