Geometrizing Characters of Tori

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Outline

Objective

K – a finite extension of $\mathbb{Q}_p$,
$\mathbf{T}$ – an algebraic torus over K (e.g. $\mathbb{G}_m$),
ℓ – a prime different from p.

Goal

Construct "geometric avatars" for characters in

$$\mathrm{Hom}(\mathbf{T}(K), \overline{\mathbb{Q}}_\ell^\times) :$$

sheaves on some space functorially associated to $\mathbf{T}$.

- Try to push characters forward along maps such as $\mathbf{T} \hookrightarrow \mathbf{G}$;
- Deligne-Lusztig representations $\Longrightarrow$ character sheaves;
- Give a new perspective on class field theory.

Approach

1. Generalize the notion of character sheaf from connected, commutative algebraic group to commutative group schemes locally of finite type.
2. Associate to $\mathbf{T}$ a projective system $\mathfrak{T}$ of commutative group schemes $\mathfrak{T}_d$ over the residue field k of K.
3. Map from character sheaves on $\mathfrak{T}$ to characters on $T(K)$.

Character Sheaves (G connected)

Two definitions of character sheaves for a connected (commutative) algebraic groups G over k:

Definition

- An *ℓ-adic local system* is a constructible sheaf of $\overline{\mathbb{Q}}_\ell$-vector spaces on the étale site of G that becomes trivial after pulling back along a finite étale map $H \to G$.
- A *geometric character sheaf* on G is an ℓ-adic local system $\mathcal{E}^\circ$ on G equipped with an isomorphism $m^*\mathcal{E}^\circ \cong \mathcal{E}^\circ \boxtimes \mathcal{E}^\circ$, where $m\colon G \times G \to G$ is multiplication.

Character Sheaves 2 (G connected)

Definition

Alternatively, a character sheaf on G is a short exact sequence

$$1 \to A \to H \to G \to 1$$

together with a character $A \to \overline{\mathbb{Q}}_\ell^\times$, so that

1. $H \to G$ is a finite étale cover,
2. Fr_q acts trivially on A.

Rationality of character sheaves

Base change to $\overline{k}$ yields a pair $(\overline{\mathcal{E}}^\circ, \mathrm{Fr}_{\mathcal{E}^\circ})$, where $\overline{\mathcal{E}}^\circ$ is a character sheaf on $\overline{G}$ and $\mathrm{Fr}_{\mathcal{E}^\circ} : \mathrm{Fr}_q^* \overline{\mathcal{E}}^\circ \xrightarrow{\sim} \overline{\mathcal{E}}^\circ$.

Proposition

In general this functor is faithful; when G is connected, base change defines an equivalence of categories

$$\left\{ \begin{array}{c} \textit{character sheaves} \\ \textit{on } G \end{array} \right\} \to \left\{ \textit{pairs } (\overline{\mathcal{E}}^\circ, \mathrm{Fr}_{\mathcal{E}^\circ}) \right\}$$

Characters in the connected case

- Suppose $(\overline{\mathcal{E}}^{\circ}, \mathrm{Fr}_{\mathcal{E}^{\circ}})$ is a character sheaf on G. Define a character $\chi_{\mathcal{E}}^{\circ}$ of $G(k)$ by

$$\chi_{\mathcal{E}}^{\circ}(x) = \mathrm{Tr}(\mathrm{Fr}_{\mathcal{E}^{\circ}}, \overline{\mathcal{E}}_x^{\circ})$$

  for $x \in G(k)$.
- Suppose χ is a character of $G(k)$. Define a character sheaf on G using the Lang isogeny $L(x) = x^{-1}\,\mathrm{Fr}_q(x)$,

$$1 \to G(k) \to G \xrightarrow{L} G \to 1,$$

  together with the character χ of $G(k)$.

Theorem (Deligne, SGA 4.5)

The maps defined above are mutually inverse isomorphisms between character sheaves on G and $\mathrm{Hom}(G(k), \overline{\mathbb{Q}}_\ell^\times)$.

Character Sheaves (G non-connected)

Definition

A character sheaf on G is a triple $\mathcal{E} = (\bar{\mathcal{E}}, \mu, F)$, where

1. $\bar{\mathcal{E}}$ is a constructible ℓ-adic sheaf on $\bar{G}$, locally constant of rank 1 on each connected component;
2. $\mu : m^*\bar{\mathcal{E}} \to \bar{\mathcal{E}} \boxtimes \bar{\mathcal{E}}$ is an isomorphism of sheaves on $\bar{G} \times \bar{G}$;
3. $F : \mathrm{Fr}_G^* \bar{\mathcal{E}} \to \bar{\mathcal{E}}$ is an isomorphism of sheaves on $\bar{G}$.

μ and F are required to satisfy various compatibility diagrams.
We write $\mathcal{CS}(G)$ for the category of character sheaves on G.

Trace of Frobenius

Write $G(k)^*$ for $\mathrm{Hom}(G(k), \overline{\mathbb{Q}}_\ell^\times)$. For any G, trace of Frobenius defines a map

$$\mathcal{CS}(G)_{/\,\mathrm{iso}} \to G(k)^*.$$

Pullback then gives a diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathcal{CS}(G/G^\circ)_{/\,\mathrm{iso}} & \longrightarrow & \mathcal{CS}(G)_{/\,\mathrm{iso}} & \longrightarrow & \mathcal{CS}(G^\circ)_{/\,\mathrm{iso}} & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & (G/G^\circ)(k)^* & \longrightarrow & G(k)^* & \longrightarrow & G^\circ(k)^* & \longrightarrow & 1
\end{array}$$

Character Sheaves 2 (G non-connected)

In the non-connected case, not every character sheaf can be realized in the second manner.

Definition

A *bounded character sheaf* on G is a short exact sequence

$$1 \to A \to H \to G \to 1$$

together with a character $A \to \overline{\mathbb{Q}}_\ell^\times$, so that

1. $H \to G$ is a finite étale cover, inducing an isomorphism on component groups
2. Fr_q acts trivially on A.

Extending character sheaves

Theorem

Every character sheaf on G° extends to a (bounded) character sheaf on G.

Proof.

Suppose that $1 \to A \to H \to G \to 1$ and $\chi \colon A \to \overline{\mathbb{Q}}_\ell^\times$ defines a bounded character sheaf. Suppose that $\mathrm{Gal}(\bar{k}/k)$ acts on H and G through the finite quotient Γ. Restriction to $H^\circ \to G^\circ$ then defines a character sheaf on G°.

Extending character sheaves

Proof.

On extension classes, this map is the first in

$$\mathrm{Ext}^1_{\mathbb{Z}[\Gamma]}(G, A) \to \mathrm{Ext}^1_{\mathbb{Z}[\Gamma]}(G^\circ, A) \to \mathrm{Ext}^2_{\mathbb{Z}[\Gamma]}(G/G^\circ, A).$$

Since $\mathbb{Z}[\Gamma]$ is a product of Dedekind domains it has cohomological dimension 1 and thus $\mathrm{Ext}^2_{\mathbb{Z}[\Gamma]}(G/G^\circ, A)$ vanishes. So $\mathrm{Ext}^1_{\mathbb{Z}[\Gamma]}(G, A) \to \mathrm{Ext}^1_{\mathbb{Z}[\Gamma]}(G^\circ, A)$ is surjective. □

Character Sheaves (G étale)

- The category of étale group schemes is equivalent to the category of groups with with Galois action.
- A character sheaf on an étale group scheme G is a collection of 1-dimensional $\overline{\mathbb{Q}}_\ell$-vectors spaces $\mathcal{E}_x$ for $x \in G(\bar{k})$ together with $F_x : \mathcal{E}_{\mathrm{Fr}(x)} \xrightarrow{\sim} \mathcal{E}_x$ and $\mu_{x,y} : \mathcal{E}_x \otimes \mathcal{E}_y \xrightarrow{\sim} \mathcal{E}_{x+y}$.

Proposition

Suppose that G is an étale commutative group scheme and $G(\bar{k})$ is finitely generated. Then there is a canonical isomorphism

$$\mathcal{CS}(G)_{/\mathrm{iso}} \cong \mathrm{H}^1(W_k, G(\bar{k})^*).$$

Trace of Frobenius Diagram

$$
\begin{array}{ccccccccc}
 & & & & & & 1 & & \\
 & & & & & & \downarrow & & \\
1 & \longrightarrow & \mathcal{CS}(G/G^{\circ})_{/\text{iso}} & \longrightarrow & \mathcal{CS}(G)_{/\text{iso}} & \longrightarrow & \mathcal{CS}(G^{\circ})_{/\text{iso}} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & (G/G^{\circ})(k)^{*} & \longrightarrow & G(k)^{*} & \longrightarrow & G^{\circ}(k)^{*} & \longrightarrow & 1 \\
 & & & & & & \downarrow & & \\
 & & & & & & 1 & &
\end{array}
$$

Trace of Frobenius (G étale)

Theorem

If G is an étale, commutative group scheme with $G(\bar{k})$ finitely generated then trace of Frobenius is an isomorphism.

Corollary

If G is a commutative group scheme with finitely generated component group then trace of Frobenius gives an isomorphism

$$\mathcal{CS}(G)_{/\,\text{iso}} \cong G(k)^*.$$

Surjectivity (G étale)

Proof.

Since $\overline{\mathbb{Q}}_\ell^\times$ is divisible it is injective as a $\mathbb{Z}$-module and thus $\mathrm{Ext}^1_{\mathbb{Z}}(G(\bar{k})/G(k), \overline{\mathbb{Q}}_\ell^\times) = 0$ so restriction

$$G(\bar{k})^* \to G(k)^*$$

is surjective. The map $\mathcal{CS}(G)_{/\mathrm{iso}} \cong \mathrm{H}^1(W_k, G(\bar{k})^*) \to G(k)^*$ is given by evaluation at Frobenius and restriction. □

Injectivity (G étale)

Proof.

Suppose $\mathcal{E}$ is in the kernel of trace-of-Frobenius, and suppose $\phi \in G(\bar{k})^*$ is the image of Frobenius under a corresponding cocycle. By assumption $\phi(x) = 1$ for $x \in G(k)$; it suffices to construct $\psi \in G(\bar{k})^*$ with $\phi(x) = \frac{\psi(x)}{\psi(\mathrm{Fr}(x))}$ for all $x \in G(\bar{k})$.

Again, let $\mathrm{Gal}(\bar{k}/k)$ act through the finite quotient Γ. Since $\mathbb{Z}[\Gamma]$ is a product of Dedekind domains, any $\mathbb{Z}[\Gamma]$-module decomposes as the direct sum of cyclic modules (generated by one element), so we may assume that $G(\bar{k})$ is a cyclic $\mathbb{Z}[\Gamma]$-module with generator y, isomorphic to $\mathbb{Z}[\Gamma]/I$.

Injectivity (G étale)

Proof.

Suppose $I = (\mathrm{Fr}^d - a_{d-1}\mathrm{Fr}^{d-1} - \cdots - a_0)$. Choose $\alpha \in \overline{\mathbb{Q}}_\ell^\times$ with

$$\phi(\mathrm{Fr}^{d-1}(y)) = \frac{\prod_{i=0}^{d-1}\left(\alpha\prod_{j=0}^{i-1}\phi(\mathrm{Fr}^j(y))\right)^{a_i}}{\alpha\prod_{j=0}^{d-2}\phi(\mathrm{Fr}^j(y))}$$

Define

$$\psi(\mathrm{Fr}^i(y)) = \alpha\prod_{j=0}^{i-1}\phi(\mathrm{Fr}^j(y))$$

for $0 \leqslant i < d$ and extend by linearity to all of $G(\bar{k})$. We have $\phi(x) = \frac{\psi(x)}{\psi(\mathrm{Fr}(x))}$ for $x = \mathrm{Fr}^i(y)$ and thus combinations.
And if I is non-principal.... □

The Néron model of a torus

R – ring of integers of K with uniformizer π
R_d – $R/\pi^{d+1}R$
$\mathbf{T}_R$ – The Néron model of $\mathbf{T}$: a separated, smooth commutative group scheme over R, locally of finite type with the Néron mapping property.

$$\mathbf{T}_R(R) = \mathbf{T}(K)$$

In the $\mathbb{G}_m$ case the Néron model is a union of copies of $\mathbb{G}_m/R$, glued along the generic fiber.
$\mathbf{T}_d$ – $\mathbf{T}_R \times_R R_d$.

Components

- The geometric component group of $\mathbf{T}_R$ fits in a sequence

$$1 \to \mathrm{H}^1(\mathcal{I}_K, X^*(\mathbf{T}))^* \to \pi_0(\mathbf{T}_R) \to \mathrm{Hom}(X^*(\mathbf{T})^{\mathcal{I}_K}, \mathbb{Z}) \to 1.$$

- $\pi_0(\mathbf{T}_R)$ is a constant group scheme after base change to the maximal unramified extension of K, but Frobenius may act nontrivially.
- The sequence of commutative R-group schemes

$$1 \to \mathbf{T}_R^\circ \to \mathbf{T}_R \to \pi_0(\mathbf{T}_R) \to 1.$$

The Greenberg functor

The Greenberg functor Gr takes a group scheme over an Artinian local ring A (locally of finite type) and produces a group scheme over the residue field k whose k points are canonically identified with the A-points of the original scheme. We set

$$\mathfrak{T}_d = \mathrm{Gr}(\mathbf{T}_d)$$

and

$$\mathfrak{T} = \varprojlim \mathfrak{T}_d.$$

$\mathfrak{T}$ is a commutative group scheme over k with

$$\mathfrak{T}(k) = \mathbf{T}(K).$$

Character sheaves on $\mathfrak{T}$

We write $\mathcal{CS}(\mathfrak{T})$ for the projective limit of the categories $\mathcal{CS}(\mathfrak{T}_d)$.

Theorem

$$T(K)^* \cong \mathcal{CS}(\mathfrak{T})_{/\,\mathrm{iso}}$$

and this isomorphism preserves depth.

Local class field theory

Suppose that L/K is a totally ramified abelian extension of local fields and we're given a character of $\mathrm{Gal}(L/K)$. The Artin reciprocity map gives a character of $K^\times$ vanishing on $\mathrm{Nm}_{L/K}(L^\times)$. We'd like to give a different description of this map, passing through character sheaves. Let $\mathbf{T} = \mathbb{G}_m$ and $\mathfrak{T}$ the Greenberg transform of $\mathbf{T}_R$.

An Isogeny

U_K – the connected Néron model of $\mathbb{G}_m$.
U_L – the connected Néron model of $\operatorname{Res}_{L/K} \mathbb{G}_m$.
H – the kernel of $\mathrm{Nm}_{L/K} \colon U_L \to U_K$.
H_0 – the subgroup of H generated by $\frac{\sigma(u)}{u}$ for $\sigma \in \mathrm{Gal}(L/K)$ and $u \in U_L$.

$$
\begin{array}{ccccccccc}
 & & H_0 & & H_0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & H & \longrightarrow & U_L & \longrightarrow & U_K & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & H/H_0 & \longrightarrow & U_L/H_0 & \longrightarrow & U_K & \longrightarrow & 1
\end{array}
$$

A Character of $\mathcal{O}_K^\times$

The Greenberg transform is exact on commutative algebraic groups, so we get a finite étale cover of $\mathfrak{T}^\circ$. Write $\mathfrak{T}_L^\circ$ for the Greenberg transform of U_L/H_0, and note that $H/H_0 \cong \mathrm{Gal}(L/K)$. Then the sequence

$$1 \to \mathrm{Gal}(L/K) \to \mathfrak{T}_L^\circ \to \mathfrak{T}^\circ \to 1,$$

together with a character of $\mathrm{Gal}(L/K)$, yields a character sheaf on $\mathfrak{T}^\circ$. From this character sheaf, we can recover a character of $\mathcal{O}_K^\times$.

Local Langlands

$\mathbf{G}$ – connected quasi-split reductive group over K
E – splitting field of $\mathbf{G}$
$\hat{\mathbf{G}}$ – dual group over $\overline{\mathbb{Q}}_\ell$
${}^L\mathbf{G}$ – $\hat{\mathbf{G}} \rtimes \mathrm{Gal}(E/K)$
φ – a tame discrete Langlands parameter $W_K \to {}^L\mathbf{G}$

A construction of DeBacker and Reeder produces from φ an unramified anisotropic torus $\mathbf{T}$ in $\mathbf{G}$ and a depth 0 character χ of $\mathbf{T}(K)$. They then describe supercuspidal representations of $\mathbf{G}(K)$ as compact inductions of Deligne-Lusztig representations determined by $\mathbf{T}$ and χ.

Geometrizing Local Langlands

In contrast to the Néron model of $\mathbf{T}$, there's no canonical integral model of $\mathbf{G}$. Instead there are many models, parameterized by the Bruhat-Tits building of $\mathbf{G}$. We hope to obtain "representation sheaves" on the Greenberg transforms of these models from character sheaves on $\mathfrak{T}$ by an analogue of Lusztig induction. Ideally, this process would allow

- the generalization of DeBacker and Reeder's methods beyond the depth 0 case,
- better understanding of the functoriality of the local Langlands correspondence,
- new descriptions of L-packets.

Clifton and I are currently pursuing these questions.

Thank you.