A Tensor Spectral Approach to Learning Mixed Membership Community Models

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Community Models in Social Networks

Social Network Modeling

- Community: group of individuals
- Community formation models: how people form communities and networks
- Community detection: Discovering hidden communities from observed network
Social Network Modeling

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Modeling Overlapping Communities

- People belong to multiple communities
- Challenging to model and learn such overlapping communities
Stochastic Block Model: Classical Approach

Generative Model

- $k$ communities and network size $n$
- Each node belongs to one community: $\pi_u = e_i$ if node $u$ is in community $i$.
- $e_i$ is the basis vector in $i^{th}$ coordinate.
- Probability of an edge from $u$ to $v$ is $\pi_u^T P \pi_v$.
- Notice that $\pi_u^T P \pi_v = P_{i,j}$ if $\pi_u = e_i$ and $\pi_v = e_j$.
- Independent Bernoulli draws for edges.

**Pros:** Guaranteed algorithms for learning block models, e.g. spectral clustering, $d_2$ distance based thresholding

**Cons:** Too simplistic. Cannot handle individuals in multiple communities
Mixed Membership Block Model

Generative Model

- $k$: communities and network size $n$
- Nodes in multiple communities: for node $u$, $\pi_u$ is community membership vector
- Probability of an edge from $u$ to $v$ is $\pi_u^\top P \pi_v$, where $P$ is block connectivity matrix
- Independent Bernoulli draws for edges

Dirichlet Priors

- Each $\pi_u$ drawn independently from $\text{Dir}(\alpha)$:
  \[
  \mathbb{P}[\pi_u] \propto \prod_{j=1}^{k} \pi_u(j)^{\alpha_j - 1}
  \]
- Stochastic block model: special case when $\alpha_j \to 0$.
- Sparse regime: $\alpha_j < 1$ for $j \in [k]$.

Learning Mixed Membership Models

Advantages

- Mixed membership models incorporate overlapping communities
- Stochastic block model is a special case
- Model sparse community membership

Challenges in Learning Mixed Membership Models

- Not clear if guaranteed learning can be provided.
- Potentially large sample and computational complexities
- Identifiability: when can parameters be estimated?
Advantages

- Mixed membership models incorporate overlapping communities
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Challenges in Learning Mixed Membership Models

- Not clear if guaranteed learning can be provided.
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Solution: Method of Moments Approach
Method of Moments

- **Inverse moment method**: solve equations relating parameters to observed moments
- **Spectral approach**: reduce equation solving to computing the “spectrum” of the observed moments
- **Non-convex** but computationally tractable approaches
Method of Moments

- Inverse moment method: solve equations relating parameters to observed moments
- Spectral approach: reduce equation solving to computing the “spectrum” of the observed moments
- Non-convex but computationally tractable approaches

Spectral Approach to Learning Mixed Membership Models

- Edge and Subgraph Counts: Moments in a network
- Tensor Spectral Approach: Low rank tensor form and efficient decomposition methods
Summary of Results and Technical Approach

Contributions

- First guaranteed learning algorithm for overlapping community models
- Correctness under exact moments.
- Explicit sample complexity bounds.
- Results are tight for Stochastic Block Models
Summary of Results and Technical Approach

Contributions

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Approach

- **Method of moments**: edge counts and 3-star count tensors
- **Tensor decomposition**: Obtain spectral decomposition of the tensor
- **Tensor spectral clustering**: Project nodes on the obtained eigenvectors and cluster.
Related Work

Stochastic Block Models
- Classical approach to modeling communities (White et al. ‘76, Fienberg et al. ‘85)
- Spectral clustering algorithm (McSherry ‘01, Dasgupta ‘04)
- $d_2$-distance based clustering (Frieze and Kannan ‘98)
  - weak regularity lemma: any dense convergent graph can be fitted to a block model

Random graph models based on subgraph counts
- Exponential random graph models
- NP-hard in general to learn and infer these models

Overlapping community models
Many empirical works but no guaranteed learning
Outline

1. Introduction

2. Tensor Form of Subgraph Counts
   - Connection to Topic Models
   - Tensor Forms for Network Models

3. Tensor Spectral Method for Learning
   - Tensor Preliminaries
   - Spectral Decomposition: Tensor Power Method

4. Conclusion
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Connection to LDA Topic Models

Exchangeable Topic Models

- $l$ words in a document $x_1, \ldots, x_l$.
- Document: topic mixture (draw of $h$).
- Word $x_i$ generated from topic $y_i$.
- Exchangeability: $x_1 \perp \perp x_2 \perp \perp \ldots | h$
- LDA: $h \sim \text{Dir}(\alpha)$.
- Learning from bigrams and trigrams

Viewing Community Models as Topic Models

- Analogy for community model: each person can function both as a document and a word.
- Outgoing links from a node $u$: node $u$ is a document.
- Incoming links to a node $v$: node $v$ is a word.
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Moments for Spectral Method

- Subgraph counts as moments of a random graph distribution

Edge Count Matrix

- Consider partition $X, A, B, C$.
- Adjacency Submatrices $G_{X,A}$, $G_{X,B}$, $G_{X,C}$
Moments for Spectral Method

- Subgraph counts as moments of a random graph distribution

### Edge Count Matrix
- Consider partition $X, A, B, C$.
- Adjacency Submatrices $G_{X,A}, G_{X,B}, G_{X,C}$

### 3-Star Count Tensor
- \# of 3-star subgraphs from $X$ to $A, B, C$.

$$M_3(u, v, w) := \frac{1}{|X|} \# \text{ of 3-stars with leaves } u, v, w$$

- Nodes in $A, B, C$: words and $X$: documents.
Moments for Spectral Method

- Subgraph counts as moments of a random graph distribution

**Edge Count Matrix**
- Consider partition $X, A, B, C$.
- Adjacency Submatrices $G_{X,A}$, $G_{X,B}$, $G_{X,C}$

**3-Star Count Tensor**
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- Nodes in $A, B, C$: words and $X$: documents.

Learning via Edge and 3-Star Counts
Recall Stochastic Block Model.

- $k$ communities and network size $n$
- Each node belongs to one community: for node $u$, $\pi_u = e_i$ if $u$ is in community $i$. $e_i$ is the basis vector in $i^{th}$ coordinate.
- Probability of an edge from $u$ to $v$ is $\pi_u \top P \pi_v$, where $P$ is block connectivity matrix.
- Independent Bernoulli draws for edges.
- Probability of edges from $X$ to $A$ is $\Pi_X \top P \Pi_A$, where $\Pi_A$ has $\pi_a$, $a \in A$ as column vectors.
- Denote $F_A := \Pi_A \top P \top$ and $\lambda_i = \mathbb{P}[\pi = e_i]$. 
Moments for Stochastic Block Model

Denote $F_A := \Pi_A^T P^T$ and $\lambda_i = \mathbb{P}[\pi = e_i]$.

Edge Count Matrix

- Adjacency Submatrices $G_{X,A}$, $G_{X,B}$, $G_{X,C}$

\[
\mathbb{E}[G_{X,A}^T \Pi_A, X] = \Pi_X^T P \Pi_A = \Pi_A^T P^T \Pi_X = F_A \Pi_X
\]
Moments for Stochastic Block Model

- Denote $F_A := \Pi_A^T \mathbf{P}^T$ and $\lambda_i = \mathbb{P}[\pi = e_i]$.

**Edge Count Matrix**

- Adjacency Submatrices $G_{X,A}$, $G_{X,B}$, $G_{X,C}$

$$\mathbb{E}[G_{X,A}^\top | \Pi_{A,X}] = \Pi_X^\top \mathbf{P} \Pi_A = \Pi_A^\top \mathbf{P}^\top \Pi_X = F_A \Pi_X$$

**3-Star Count Tensor**

- # of 3-star subgraphs from $X$ to $A$, $B$, $C$.

$$M_3 := \frac{1}{|X|} \sum_{i \in X} [G_{i,A}^\top \otimes G_{i,B}^\top \otimes G_{i,C}^\top]$$

$$\mathbb{E}[M_3 | \Pi_{A,B,C}] = \sum_i \lambda_i [(F_A)_i \otimes (F_B)_i \otimes (F_C)_i]$$

**Goal:** Recover $F_A, F_B, F_C, \mathbf{\lambda}$
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Tensor Basics: Multilinear Transformations

- For a tensor $T$, define (for matrices $V_i$ of appropriate dimensions)
  \[
  [T(W_1, W_2, W_3)]_{i_1,i_2,i_3} := \sum_{j_1,j_2,j_3} (T)_{j_1,j_2,j_3} \prod_{m\in[3]} W_m(j_m, i_m)
  \]

- For a matrix $M$, \[M(W_1, W_2) := W_1^\top MW_2\].

- For a symmetric tensor $T$ of the form
  \[T = \sum_{r=1}^{k} \lambda_r \phi_r \otimes 3\]
  \[
  T(W, W, W) = \sum_{r\in[k]} \lambda_r (W^\top \phi_r) \otimes 3
  \]
  \[
  T(I, v, v) = \sum_{r\in[k]} \lambda_r \langle v, \phi_r \rangle^2 \phi_r.
  \]
  \[
  T(I, I, v) = \sum_{r\in[k]} \lambda_r \langle v, \phi_r \rangle \phi_r \phi_r^\top.
  \]
Whiten: Convert to Orthogonal Symmetric Tensor

- Assume exact moments are known.

\[
\mathbb{E}[G_{X,A}^\top | \Pi_{A,X}] = F_A \Pi_X
\]

\[
\mathbb{E}[M_3 | \Pi_{A,B,C}] = \sum_i \lambda_i [(F_A)_i \otimes (F_B)_i \otimes (F_C)_i]
\]
Whiten: Convert to Orthogonal Symmetric Tensor

- Assume exact moments are known.
  \[ \mathbb{E}[G_{X,A}^{\top}|\Pi_{A,X}] = F_A \Pi_X \]
  \[ \mathbb{E}[M_3|\Pi_{A,B,C}] = \sum_i \lambda_i [(F_A)_i \otimes (F_B)_i \otimes (F_C)_i] \]

- Use SVD of \( G_{X,A}, G_{X,B}, G_{X,C} \) to obtain whitening matrices \( W_A, W_B, W_C \)

- Apply multi-linear transformation on \( M_3 \) using \( W_A, W_B, W_C \).
  \[ T := \mathbb{E}[M_3(W_A, W_B, W_C)|\Pi_{A,B,C}] = \sum_i w_i \mu_i^{\otimes 3} \]

- \( T \) is symmetric orthogonal tensor: \( \{\mu_i\} \) are orthonormal.

Spectral Tensor Decomposition of \( T \)
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Orthogonal Tensor Eigen Analysis

Consider orthogonal symmetric tensor $T = \sum_{i} w_i \mu_i \otimes^3$

$$T = \sum_{i=1}^{k} w_i \mu_i \otimes^3. \quad T(I, \mu_i, \mu_i) = w_i \mu_i$$
Consider orthogonal symmetric tensor \( T = \sum_i w_i \mu_i \otimes^3 \)

\[
T = \sum_{i=1}^k w_i \mu_i \otimes^3. \quad T(I, \mu_i, \mu_i) = w_i \mu_i
\]

Obtaining eigenvectors through power iterations

\[
u \mapsto \frac{T(I, u, u)}{\|T(I, u, u)\|}
\]
Orthogonal Tensor Eigen Analysis

- Consider orthogonal symmetric tensor $T = \sum_i w_i \mu_i \otimes^3$

$$T = \sum_{i=1}^{k} w_i \mu_i \otimes^3. \quad T(I, \mu_i, \mu_i) = w_i \mu_i$$

**Obtaining eigenvectors through power iterations**

$$u \mapsto \frac{T(I, u, u)}{\|T(I, u, u)\|}$$

**Challenges and Solution**

- **Challenge:** Other eigenvectors present

  **Solution:** Only stable vectors are basis vectors $\{\mu_i\}$
Orthogonal Tensor Eigen Analysis

- Consider orthogonal symmetric tensor \( T = \sum_i w_i \mu_i \otimes^3 \)

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T = \sum_{i=1}^{k} w_i \mu_i \otimes^3. \quad T(I, \mu_i, \mu_i) = w_i \mu_i
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Obtaining eigenvectors through power iterations

\[
\mathbf{u} \mapsto \frac{T(I, \mathbf{u}, \mathbf{u})}{\|T(I, \mathbf{u}, \mathbf{u})\|}
\]

Challenges and Solution

- Challenge: Other eigenvectors present
  Solution: Only stable vectors are basis vectors \( \{\mu_i\} \)

- Challenge: empirical moments
  Solution: robust tensor decomposition methods
Optimization Viewpoint for Tensor Eigen Analysis

Consider Norm Optimization Problem for Tensor $T$

\[
\max_u T(u, u, u) \quad \text{s.t. } u^\top u = I
\]

- Constrained stationary fixed points $T(I, u, u) = \lambda u$ and $u^\top u = I$.
- $u$ is a local isolated maximizer if $w^\top (T(I, I, u) - \lambda I)w < 0$ for all $w$ such that $w^\top w = I$ and $w$ is orthogonal to $u$.

Review for Symmetric Matrices $M = \sum_i w_i \mu_i \otimes 2$

- Constrained stationary points are the eigenvectors
- Only top eigenvector is a maximizer and stable under power iterations

Orthogonal Symmetric Tensors $T = \sum_i w_i \mu_i \otimes 3$

- Stationary points are the eigenvectors (up to scaling)
- All basis vectors $\{\mu_i\}$ are local maximizers and stable under power iterations
Tensor Decomposition: Perturbation Analysis

- Observed tensor $\tilde{T} = T + E$, where $T = \sum_{i \in k} w_i \mu_i \otimes^3$ is orthogonal tensor and perturbation $E$, and $\|E\| \leq \epsilon$.

- Recall power iterations $u \mapsto \frac{\tilde{T}(I, u, u)}{\|\tilde{T}(I, u, u)\|}$
Tensor Decomposition: Perturbation Analysis

- Observed tensor $\tilde{T} = T + E$, where $T = \sum_{i \in k} w_i \mu_i \otimes^3$ is orthogonal tensor and perturbation $E$, and $\|E\| \leq \epsilon$.

- Recall power iterations $u \mapsto \frac{\tilde{T}(I, u, u)}{\|\tilde{T}(I, u, u)\|}$

- “Good” initialization vector $\langle u^{(0)}, \mu_i \rangle^2 = \Omega \left( \frac{\epsilon}{w_{\min}} \right)$
Tensor Decomposition: Perturbation Analysis

- Observed tensor $\tilde{T} = T + E$, where $T = \sum_{i \in k} w_i \mu_i \otimes 3$ is orthogonal tensor and perturbation $E$, and $\|E\| \leq \epsilon$.

- Recall power iterations

$$ u \mapsto \frac{\tilde{T}(I, u, u)}{\|\tilde{T}(I, u, u)\|} $$

- “Good” initialization vector

$$ \langle u^{(0)}, \mu_i \rangle^2 = \Omega \left( \frac{\epsilon}{w_{\min}} \right) $$

Perturbation Analysis

After $N$ iterations, eigen pair $(w_i, \mu_i)$ is estimated up to $O(\epsilon)$ error, where

$$ N = O \left( \log k + \log \log \frac{w_{\max}}{\epsilon} \right) $$

Robust Tensor Power Method

\[ \tilde{T} = \sum_i w_i \mu_i \otimes^3 + E \]

Basic Algorithm

- Pick random initialization vectors
- Run power iterations
  \[ u \mapsto \frac{\tilde{T}(I, u, u)}{\|\tilde{T}(I, u, u)\|} \]
- Go with the winner, deflate and repeat
Robust Tensor Power Method

\[ \widetilde{T} = \sum_i w_i \mu_i^\otimes 3 + E \]

Basic Algorithm

- Pick random initialization vectors
- Run power iterations
  \[ u \mapsto \frac{\widetilde{T}(I, u, u)}{\|\widetilde{T}(I, u, u)\|} \]
- Go with the winner, deflate and repeat

Further Improvements

- Initialization: Use neighborhood vectors for initialization
- Stabilization:
  \[ u^{(t)} \mapsto \alpha \frac{\widetilde{T}(I, u^{(t-1)}, u^{(t-1)})}{\|\widetilde{T}(I, u^{(t-1)}, u^{(t-1)})\|} + (1 - \alpha)u^{(t-1)} \]

Efficient Learning Through Tensor Power Iterations
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Mixed Membership Models
- Can model overlapping communities
- Efficient to learn from low order moments: edge counts and 3-star counts.

Tensor Spectral Method
- Whitened 3-star count tensor is an orthogonal symmetric tensor
- Efficient decomposition through power method
- Perturbation analysis: tight for stochastic block model