# A Conjecture on the Distribution of Balanced Load in a Large Network

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# Abstract:

This note recalls a conjecture appearing in [1] about the distribution of balanced load in a large network.

# The load balancing model

- Undirected multigraph with $M$ nodes (locations of a resource)
- $\alpha M$ edges (representing consumers)
- One unit of load associated with each edge
- Load of an edge is to be assigned to endpoints, splitting allowed
- Load is *balanced* if for any edge incident on $u$ and $v$, if the load at $u$ is strictly larger than the load at $v$, the entire load of the edge must be assigned to $v$.

# Illustration of a balanced load

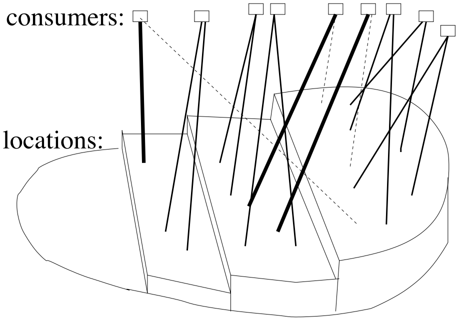


Figure: Illustration of a balanced load.

# Random graph model

- The endpoints of each of the $\alpha M$ edges are selected independently, uniformly at random, from among all $\binom{M}{2}$ pairs of nodes.
- Tables 1 and 2 below show the results of a simulation for $M = 10000$ and $\alpha = 2$.
- For Table 1 the load of each edge is split evenly
- For Table 2 the loads are balanced.
- Let $F(\tau; \alpha)$ denote the CDF of limiting load distribution as $M \to \infty$.

Table: 1. Sample load distribution *before* balancing ($\alpha = 2, M = 10000$).

| $\tau$ | load$\leq \tau$ | load$= \tau$ |
|---|---|---|
| 0.0 | 201 | 201 |
| 0.5 | 921 | 720 |
| 1.0 | 2382 | 1461 |
| 1.5 | 4299 | 1917 |
| 2.0 | 6291 | 1992 |
| 2.5 | 7896 | 1605 |
| 3.0 | 8899 | 1003 |
| 3.5 | 9472 | 573 |
| 4.0 | 9778 | 306 |
| 4.5 | 9912 | 134 |
| 5.0 | 9962 | 50 |
| 5.5 | 9987 | 25 |
| 6.0 | 10000 | 13 |

Table: 2. Sample load distribution *after* balancing ($\alpha = 2, M = 10000$).

| $\tau$ | load$\leq \tau$ | load$= \tau$ | product |
|---|---|---|---|
| 0.00000000 | 201 | 201 | 0 |
| 0.50000000 | 223 | 22 | 11 |
| 1.00000000 | 992 | 769 | 769 |
| 1.25000000 | 996 | 4 | 5 |
| 1.33333333 | 1023 | 27 | 36 |
| 1.50000000 | 1239 | 216 | 324 |
| 1.60000000 | 1244 | 5 | 8 |
| 1.66666667 | 1313 | 69 | 115 |
| 1.75000000 | 1353 | 40 | 70 |
| 1.77777778 | 1362 | 9 | 16 |
| 1.80000000 | 1392 | 30 | 54 |
| 1.83333333 | 1398 | 6 | 11 |
| 1.85714286 | 1405 | 7 | 13 |
| 1.92307692 | 1418 | 13 | 25 |
| 2.00000000 | 3316 | 1898 | 3796 |
| 2.07692308 | 3329 | 13 | 27 |
| 2.11111111 | 3338 | 9 | 19 |
| 2.12500000 | 3362 | 24 | 51 |
| 2.14285714 | 3404 | 42 | 90 |
| 2.16666667 | 3440 | 36 | 78 |
| 2.18181818 | 3462 | 22 | 48 |
| 2.20000000 | 3562 | 100 | 220 |
| 2.20782852 | 10000 | 6438 | 14214 |

# Poisson tree network

- The mean number of edges incident on a given node is $\lambda = 2\alpha$.
- For $M$ large the neighborhood of a fixed node $v_o$ becomes like a Poisson($2\alpha$) tree network rooted at $v_o$.
- If the loads of the tree network at distance $k$ from $v_o$ are clamped at a specific value, the distribution of the load at $v_o$ (after balancing) can be computed [1].
- Let $F_T(\tau; 2\alpha)$ denote the limiting load distribution obtained as $k \to \infty$.
- Numerical values of $F_T(\tau; 2\alpha)$ for $\alpha = 2$ are shown in Table 3.

## Table: 3. Numerical calculations of $F_T(\tau, 2\alpha)$ for $\alpha = 2$.

| $\tau$ | $a(\tau; 2\alpha)$ | $F_T(\tau; 2\alpha)$ | $\int_0^\tau 1 - F_T(u; 2\alpha)du$ |
|---|---|---|---|
| 0 | 0.01832 | 0.01832 | 0 |
| 0.0833 | 0.00000 | 0.01832 | 0.08181 − 0.08181 |
| 0.1667 | 0.00000 | 0.01832 | 0.16361 − 0.16361 |
| 0.2500 | 0.00000 | 0.01832 | 0.24542 − 0.24542 |
| 0.3333 | 0.00000 | 0.01832 | 0.32723 − 0.32723 |
| 0.4167 | 0.00000 | 0.01832 | 0.40904 − 0.40904 |
| 0.5000 | 0.00134 | 0.01966 | 0.49084 − 0.49084 |
| 0.5833 | 0.00000 | 0.01966 | 0.57254 − 0.57254 |
| 0.6667 | 0.00015 | 0.01980 | 0.65423 − 0.65423 |
| 0.7500 | 0.00002 | 0.01982 | 0.73592 − 0.73592 |
| 0.8333 | 0.00000 | 0.01983 | 0.81760 − 0.81760 |
| 0.9167 | 0.00000 | 0.01983 | 0.89928 − 0.89928 |
| 1.0000 | 0.07774 | 0.09756 | 0.98096 − 0.98096 |
| 1.0833 | 0.00000 | 0.09756 | 1.05616 − 1.05616 |
| 1.1667 | 0.00000 | 0.09757 | 1.13136 − 1.13136 |
| 1.2500 | 0.00026 | 0.09785 | 1.20657 − 1.20657 |
| 1.3333 | 0.00246 | 0.10031 | 1.28174 − 1.28175 |
| 1.4167 | 0.00000 | 0.10050 | 1.35670 − 1.35672 |
| 1.5000 | 0.02180 | 0.12232 | 1.43166 − 1.43168 |
| 1.5833 | 0.00000 | 0.12240 | 1.50479 − 1.50482 |
| 1.6667 | 0.00920 | 0.13236 | 1.57786 − 1.57795 |
| 1.7500 | 0.00462 | 0.13728 | 1.65014 − 1.65026 |
| 1.8333 | 0.00150 | 0.14151 | 1.72181 − 1.72215 |
| 1.9167 | 0.00012 | 0.14399 | 1.79315 − 1.79369 |
| 2.0000 | 0.19107 | 0.33533 | 1.86446 − 1.86502 |
| 2.0833 | 0.00024 | 0.33598 | 1.91982 − 1.92041 |
| 2.1667 | 0.00554 | 0.34945 | 1.97449 − 1.97575 |
| 2.2500 | 0.02083 | 0.38899 | 2.02715 − 2.02996 |
| 2.3333 | 0.04704 | 0.47013 | 2.07522 − 2.08088 |
| 2.4167 | 0.00000 | 1.00000 | 2.07522 − 2.12503 |
| 2.5000 | 0.00000 | 1.00000 | 2.07522 − 2.12503 |

# A comparison plot

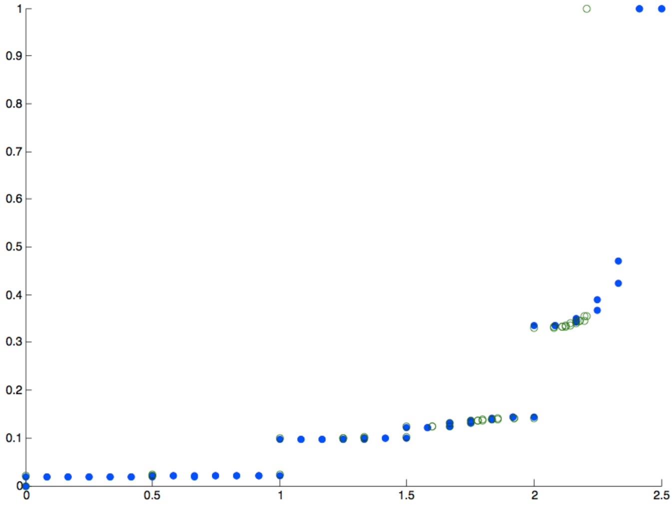


Figure: Comparison of $F(\tau; \alpha, M)$ (open circles) and $F_T(\tau, 2\alpha)$ (filled circles) for $\alpha = 2$ and $M = 10000$.

# Matching the mean

- Load at fixed node in limit $M \to \infty$ is less than limiting load for infinite tree network with load at boundary clamped at infinity: $F(\tau;\alpha) \geq F_T(\tau;2\alpha)$ for all $\tau \geq 0$.
- Mean for CDF $F(\tau;\alpha)$ is $2\alpha : \alpha = \int_0^\infty 1 - F(\tau;2\alpha)d\tau$
- But $\int_0^\infty 1 - F_T(\tau;2\alpha)d\tau > \alpha$

# Truncation of distribution

Let $\triangle(\alpha)$ be defined by $\alpha = \int_0^{\triangle(\alpha)} 1 - F_T(\tau; 2\alpha) d\tau$

Table: Some numerical values of $\triangle(\alpha)$ and $1 - F_T(\triangle(\alpha)-; 2\alpha)$.

| $\alpha$ | $\triangle(\alpha)$ | $1 - F_T(\triangle(\alpha)-; 2\alpha)$ |
|---|---|---|
| 0.2 | 1 | 0 |
| 0.4 | 1 | 0 |
| 0.6 | 1.07235 – 1.08333 | 0.00000 – 0.03520 |
| 0.8 | 1.22226 – 1.22794 | 0.11913 – 0.14205 |
| 1.0 | 1.37544 – 1.37757 | 0.25944 – 0.26825 |
| 1.2 | 1.52653 – 1.53595 | 0.35094 – 0.37209 |
| 1.4 | 1.68945 – 1.69542 | 0.46912 – 0.49171 |
| 1.6 | 1.84892 – 1.85339 | 0.58498 – 0.60966 |
| 1.8 | 2.00525 – 2.00764 | 0.50888 – 0.50975 |
| 2.0 | 2.20438 – 2.20593 | 0.63826 – 0.65055 |
| 4.0 | 4.07933 – 4.07937 | 0.87073 – 0.87074 |
| 6.0 | 6.03571 – 6.03575 | 0.94613 – 0.94615 |
| 8.0 | 8.01720 – 8.01721 | 0.97578 – 0.97578 |
| 10.0 | 10.00857 – 10.00857 | 0.98855 – 0.98855 |

# Giant component

- Moment method shows that for $\alpha$ not too small, as $M \to \infty$, there is a giant component of the graph with all nodes in the component having the maximum load.
- Suggests that load at leaves of Poisson tree be clamped to maximum load expected, $\triangle(\alpha)$.

# The conjecture

**Conjecture.** For all $\tau, \alpha \geq 0$,

$$\lim_{M \to \infty} F(\tau; M, \alpha) = \begin{cases} F_T(\tau; 2\alpha) & \text{if } \tau < \triangle(\alpha) \\ 1 & \text{if } \tau \geq \triangle(\alpha). \end{cases} \qquad (1)$$

Furthermore,

$$\lim - \text{in} - \text{prob.}_{M \to \infty} \frac{|\widehat{A}|}{M} = 1 - F_T(\triangle(\alpha)-; 2\alpha).$$

The conjecture is true if either $0 \leq \tau < 1$ or $0 \leq \alpha \leq 0.5$.

B. Hajek, "Performance of global load balancing by local adjustment," *IEEE Trans. Information Theory*, Vol. 36, Nov. 1990. pp. 1398 - 1414.