WIT: Women in Topology

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August 18 - August 25, 2013

Women in Topology (Banff 2013) was the first workshop organized by our newly founded network WIT-HT (Women in Topology - Homotopy Theory) as an effort to increase the number and visibility of active female researchers in homotopy theory, as well as the participation by women in research activities in the field. The main feature of the meeting was engagement in collaborative group projects for teams of 5-7 participants, each including senior and junior researchers, as well as students.

In addition to the generous support of the Banff International Research Station, we would also like to thank the Clay Mathematics Institute's Enhancement and Partnership program, which funded travel for our participants, and the Pacific Institute for Mathematical Sciences and École Polytechnique Fédérale de Lausane for additional support.

1 Objectives

The goal of our workshop was to support and expand the research efforts by female mathematicians in the field of homotopy theory. Inspired by the success of the Women in Numbers network (who held 5-day BIRS Workshops in 2008 and 2011, with another to follow in 2014), we shared similar objectives. In particular, we aimed to contribute to the success of women in homotopy theory by

- contributing to the training of graduate students;
- highlighting the research activities of women in homotopy theory;
- building collaborations amongst female researchers in homotopy theory;
- increasing the participation of women in research activities in homotopy theory; and
- publishing the findings of each team.

We believe that we have achieved the first four goals. We have also made significant progress on the final goal, as each team has submitted a research paper on their results to a volume of Contemporary Mathematics dedicated to the workshop. Some of the teams have continued the collaborations that began at the WIT workshop.

When surveyed afterwards, the workshop participants reported overwhelmingly that the workshop had a significant positive impact on them as individuals. Testimonials from this first workshop can be found at the link

http://www.birs.ca/events/2013/5-day-workshops/13w5145/testimonials

In summary, the workshop participants reported that the workshop had a positive impact on their research, was more productive than a usual workshop as each team produced a new result, and created new collaborations amongst the team members.

In order to best capitalize on the experience, we have created a Women in Topology - Homotopy Theory network which is being used to continue collaborations, make women aware of important opportunities in homotopy theory, and offer education and support for issues which are surmountable obstacles to success for women in mathematics.

2 Participants and Format

Participation in the WIT workshop was limited to women in order to best support our objectives. We initially solicited research project proposals from 15 established researchers (women with permanent positions) who would eventually lead 7 teams. We then began to invite team members, selecting these from graduate students, recent graduates, post doctoral fellows, and women who indicated an eagerness to increase research activity after a lapse in activity. We solicited recommendations for participants from a lengthy list of graduate student supervisors in Canada, the US and Europe. We were fortunate to be able to accommodate all of the women who indicated interest in the workshop.

After establishing our participant list, we advertised project proposals to team members. Team members were asked to rank their preference for team membership, and we used this information together with consideration of each member's expertise to form cohesive teams.

In the winter preceding the workshop, team leaders began to contact their teams to prepare and train them. Each team prepared differently. Some team leaders prepared their teams by distributing a list of required reading and prerequisites well in advance of the meeting, others sent "homework" assignments or exercises to engage their teams in advance. Every team had enough exposure to their proposed problem in advance to be able to start working when they arrived.

On the first day of the workshop, team leaders presented a very short introduction to their proposed research problem. These presentations were 5-10 minutes in length, and the purpose of these presentations was more to provide a brief description of the proposed project to non-team members, since the team members were already familiar with the project upon arrival. On the last day of the workshop, the team members (as opposed to the team leaders) provided a brief (20 minute) presentation of the team's findings. The project presentations typically involved all of the team members, thus every workshop participant had an opportunity to give a talk. The lectures were essential for making all workshop participants familiar with each other's work, and helped in establishing networks outside of the teams.

In between these events, the teams engaged in intensive research. We organized six 3 hour working sessions for the teams, but most of the teams simply filled their entire time by doing mathematics. There were two non-mathematical events scheduled (aside from meals): these were two evening discussion of issues of concern to women in mathematics.

After the workshop, we established a network of women researchers called Women in Topology - Homotopy Theory. We have continued to grow this network by inviting more participants. The network advertises opportunities in math as well as pertinent information for women working in any field in which they are underrepresented.

Finally, the teams continued to work after the workshop concluded. Each team prepared a research paper on their results. These papers are currently being refereed for a volume of Contemporary Mathematics devoted to the workshop. In Sections 3 – 9 of this document, the main results obtained by each of the teams is described. Briefly, the seven teams considered

- Bredon homology of the poset of direct-sum decompositions of $\mathcal{C}^n$;
- an investigation of derived A-infinity algebras;
- calculations of higher order topological Hochschild homology;
- calculations and examples of unbased functor calculus related to André-Quillen homology;
- the existence of a model structure on the category of small G-categories;

- the establishment of a left-induced model category structure and Postnikov presentations of model categories; and
- an investigation of the mapping spaces of orbispaces.

These projects represent diverse areas within homotopy theory, and are numerous enough to touch on most of the major active areas of research in the field. Some of the teams have continued their collaborations beyond their first project.

The final reports for each team are below. Team members marked with * are team leaders.

3 Bredon Homology: Julie Bergner*, Ruth Joachimi, Kathryn Lesh*, Vesna Stojanoska, Kirsten Wickelgren

Our project is part of a program to generalize recent work of Arone, Dwyer, and Lesh, [2], in which they compute Bredon homology and cohomology of the partition complex for the set $\{1,\ldots,n\}$, to a bu-version involving Bredon homology and cohomology for the poset of direct-sum decompositions of $\mathbb{C}^n$. One motivation for the work in the discrete case is an expected proof of the Whitehead Conjecture which does not rely on detailed homology calculations, and an analogous statement is expected to hold in the bu-case.

Arone and Lesh set up a parallel picture to the discrete case in the context of the unitary group [1]. For example, the symmetric group Σ_n corresponds to the unitary group $U(n)$, p-subgroups correspond to p-toral subgroups, and the partition complex for $\{1,\ldots,n\}$ corresponds to the partition complex $\mathcal{L}_n$ for $\mathbb{C}^n$ by orthogonal direct-sum decompositions. The first step in the unitary program, to establish the analogues of Arone, Dwyer, and Lesh's results for the bu-case, was the motivating problem for our team.

In particular, our goal was to answer the following question.

Problem 1. Characterize the p-toral subgroups H of $U(n)$ for which $\mathcal{L}_n^H$ fails to be contractible.

The first expected part of the answer to characterizing these "problematic" subgroups was that they must be projective elementary abelian p-groups of $U(n)$, and we proved that this fact is indeed true. However, it is known that this condition is not sufficient; the remainder of our work was concerned with further refining the conditions under which the $\mathcal{L}_n^H$ is not contractible. We were able to prove the following further characterizations of the groups H:

- the group H must be abstractly isomorphic to one of the form $\Gamma_i \times \Delta_j$ (where these groups are as defined by Oliver), and
- if $k = i + j$, then H is (conjugate to) a subgroup of Γ_k diagonally embedded in $U(n)$ (where $n = mp^k$ for some m).

The next step in the program, which the team has begun to consider, is the following. Let $\mathcal{A}$ be the collection of all p-toral subgroups of $U(n)$, and let Γ be a p-constrained Mackey functor for $U(n)$, in the sense of Libman, [13].

Problem 2. Establish that for any reasonable $U(n)$-space X, the approximation $X_{\mathcal{A}} \to X$ is an isomorphism on Bredon homology and cohomology with coefficients in Γ.

4 Derived A-infinity algebras: Camil Aponte, Muriel Livernet*, Marcy Robertson, Sarah Whitehouse*, Stephanie Ziegenhagen

The study of A_∞-algebras goes back to the work of Stasheff [18] on group-like topological spaces in the sixties. Since then the importance of A_∞-structures has become well established in many areas including algebra, geometry and mathematical physics. Working over a ground field, these structures play a key role in the theory of minimal models and classification of differential graded algebras (dgas) up to quasi-isomorphism.

Recently Sagave [17] developed the notion of derived A_∞-algebra, in order to have a theory of minimal models for dgas over a general commutative ring. Joint work of Livernet, Roitzheim and Whitehouse [14] introduced a description of this structure using operads. This project has developed this operadic approach to derived A_∞-structures in several directions.

Our operads are nonsymmetric operads in the category of bicomplexes with zero horizontal differential. We start from an operad $\mathrm{d}\mathcal{A}s$ in this category encoding bidgas, that is, monoids in bicomplexes. Previous work establishes that derived A_∞-algebras are precisely algebras over the operad

$$dA_\infty = (\mathrm{d}\mathcal{A}s)_\infty = \Omega((\mathrm{d}\mathcal{A}s)^{\text{!`}}).$$

Thus we can view a derived A_∞-algebra as the infinity version of a bidga, in the same sense that an A_∞-algebra is the infinity version of a dga.

The first part of the project involved investigating the operad $\mathrm{d}\mathcal{A}s$ further, in particular giving a simple description of $(\mathrm{d}\mathcal{A}s)^{\text{!`}}$-coalgebras. In the classical case, the structure of an $\mathcal{A}s^{\text{!`}}$-coalgebra is well-known to be the same as a usual coassociative coalgebra. In the case of $(\mathrm{d}\mathcal{A}s)^{\text{!`}}$-coalgebras, we obtain coassociative coalgebras with an extra piece of structure. We also study representations of derived A_∞-algebras, via suitable coderivations and also concretely in terms of comodules over the corresponding coassociative coalgebras that are suitably compatible with the extra structure. We explain how this relates to Sagave's derived A_∞-modules described in terms of coderivations on the reduced cotensor algebra.

The next part covers model category structures on the category of derived A_∞-algebras, establishing the existence of a model structure whose weak equivalences are Sagave's E_2-equivalences. Among the good properties of this structure are that Sagave's resolutions are cofibrant objects and that there is a nice relationship to a previously developed model structure on A_∞-algebras due to Lefèvre-Hasegawa [12].

Finally, we have defined some new explicit families of examples of derived A_∞-algebras. The construction is based on some examples of A_∞-algebras due to Allocca [1].

Overall, substantial progress was made on developing elements of the theory of derived A_∞-algebras. The emphasis of our work turned out to differ quite substantially from the original project proposal, which focused on cohomology of these algebras. This is a topic that we expect to return to in future work building on this project.

5 Calculations of Higher Order Topological Hochschild Homology: Irina Bobkova, Ayelet Lindenstrauss*, Kate Poirier, Birgit Richter*, Inna Zakharevich

Given a commutative ring T and a T-module M, J.-L. Loday (e.g.,[15]) introduced a functor $\mathcal{L}(T, M)$ which takes a based simplicial set $X.$ to the simplicial T-module which consists in degree n of M tensored with one copy of T with each element in $X_n \setminus \{*\}$, with face maps d_i sending the T corresponding to $\tau \in X_n$ to the T or M corresponding to $d_i(\tau) \in X_{n-1}$ (and multiplying everything that lands in the same coordinate). Applying this functor to the usual model of S^1 with one non-degenerate 0-cell and one non-degenerate 1-cell, we get the classical Hochschild complex whose homology is $\mathsf{HH}_*(T; M)$. Extending this, the higher topological Hochschild homology $\mathsf{HH}_*^{[n]}(T; M)$ was defined to be the homology of $\mathcal{L}(T, M)$ of S^n. The homology of the image of the Loday functor turns out to be independent of the simplicial structure used on $|X.|$, and moreover depends only on its homotopy type.

M. Brun, G. Carlsson, and B. Dundas [5] introduced a topological version of $\mathcal{L}(T, M)$ for a ring spectrum T and T-module M. When evaluated on S^n, it yields the spectrum $\mathsf{THH}^{[n]}(T; M)$, higher topological Hochschild homology.

T. Veen [22] used a decomposition result on $\mathcal{L}(T, M)$ to inductively calculate

$$\mathsf{THH}_*^{[n]}(\mathcal{F}_p) = \pi_*(\mathsf{THH}^{[n]}(\mathcal{F}_p))$$

for all $n \leq 2p$. (When $M = T$ with the obvious action by multiplication, it is omitted from the notation.) He sets up a spectral sequence of Hopf algebras calculating $\mathsf{THH}_*^{[n]}(\mathcal{F}_p)$ from $\mathsf{THH}_*^{[n-1]}(\mathcal{F}_p)$ (with the base case $\mathsf{THH}_*^{[1]}(\mathcal{F}_p)$ being known by work of M. Bökstedt), and explains why it has to collapse for $n \leq 2p$. By more careful analysis of the structure of the spectral sequence, motivated by computer calculations, we were able to show that Veen's spectral sequence collapses for $n \leq 2p+2$, thus getting a calculation of $\mathsf{THH}_*^{[n]}(\mathcal{F}_p)$ for those n. The computer analysis found, however, potential nontrivial differentials in the spectral sequence when $n = 2p+3$.

We were also able to show that for an $\mathcal{F}_p$-algebra A and a abelian group G,

$$\mathsf{THH}_*^{[n]}(A[G]) \cong \mathsf{THH}_*^{[n]}(A) \otimes \mathsf{HH}_*^{[n]}(\mathcal{F}_p[G]).$$

Using this, we calculated $\mathsf{THH}_*^{[n]}(\mathcal{F}_p[G])$ for any finitely generated abelian group G for $n \leq 2p+1$. (For general abelian groups, observe that higher Hochschild homology commutes with direct limits.) Since we know $\mathsf{THH}_*^{[n]}(\mathcal{F}_p)$ in that range, we only need to calculate $\mathsf{HH}_*^{[n]}(\mathcal{F}_p[G])$. But for abelian groups H and G, $\mathcal{F}_p[G \times H] \cong \mathcal{F}_p[G] \otimes \mathcal{F}_p[H]$, and therefore $\mathsf{HH}_*^{[n]}(\mathcal{F}_p[G \times H]) \mathcal{O}ng \mathsf{HH}_*^{[n]}(\mathcal{F}_p[G]) \otimes \mathsf{HH}_*^{[n]}(\mathcal{F}_p[H])$. Thus we only need to consider the case of cyclic groups G. By Veen's method, we can calculate $\mathsf{HH}_*^{[n]}(\mathcal{F}_p[C_k])$ for finite cyclic groups, and also $\mathsf{HH}_*^{[n]}(\mathcal{F}_p[x])$. Since $\mathcal{F}_p[\mathsf{Z}]$ is étale over $\mathcal{F}_p[x]$, we prove that $\mathsf{HH}_*^{[n]}(\mathcal{F}_p[\mathsf{Z}]) \cong \mathcal{F}_p[\mathsf{Z}] \otimes_{\mathcal{F}_p[x]} \mathsf{HH}_*^{[n]}(\mathcal{F}_p[x])$, and conclude the calculation.

6 Functor Calculus: Maria Basterra*, Kristine Bauer*, Agnes Beudry, Rosona Eldred, Brenda Johnson*, Mona Merling, Sarah Yeakel

In a series of papers published between 1990 and 2003, Tom Goodwillie developed what is now known as the calculus of homotopy functors [6], [7], [8]. The calculus of homotopy functors associates to a given functor of spaces or spectra F, a so-called Taylor tower of functors and natural transformations,

$$\begin{array}{ccccccccc} & & & & F & & & & \\ & & & \swarrow & \downarrow & \searrow & & & \\ \cdots \longrightarrow & P_{n+1}F & \longrightarrow & & P_nF & \longrightarrow & P_{n-1}F \cdots \longrightarrow & P_1F \longrightarrow & P_0F, \end{array}$$

resembling the Taylor series for functions of real variables. In particular, Goodwillie's theory produces a universal n-excisive approximation to a homotopy functor F. Inspired by Goodwillie's work, Brenda Johnson and Randy McCarthy [10] produced a related theory of calculus in an abelian setting which produces what can be thought of as a "discrete" Taylor tower for a functor.

In the Johnson-McCarthy discrete calculus, a homotopy functor is approximated by a universal *degree* n functor. While n-excisive functors are necessarily degree n, the converse does not generally hold. The Johnson-McCarthy model was originally developed for use in algebraic settings, and functors were assumed to be from a pointed category to an abelian category (often, chain complexes). The hypothesis that the domain category be pointed was more restrictive than what Goodwillie's theory required, nonetheless, the pointed theory has been quite useful. In particular, it has been used successfully to express certain interesting homology theories as derivatives of naturally arising functors. For example, Johnson and McCarthy [10], and Kantorovitz and McCarthy [11] have provided ways of viewing André-Quillen homology as parts of discrete calculus towers.

Recently, the Johnson-McCarthy theory of calculus was expanded by Kristine Bauer, Brenda Johnson and Randy McCarthy [3]. The new theory includes functors from categories that are not necessarily pointed to categories that are not necessarily abelian. At the *Women in Topology* workshop, our team set, as a general goal, the development of some concrete examples for this new theory of "unbased" discrete calculus. These examples went in three distinct, yet interrelated, directions.

The first line of work revisited the construction of the nth term, P_nF, in the unbased discrete Taylor tower of F, using a streamlined model of Ben Walter's for homotopy pullbacks in differential graded rational vector spaces, $\mathcal{D}G$. The result of this direction was an alternative proof of one of the key results used in [3] to construct P_nF for functors to $\mathcal{D}G$. In particular, the functor P_nF relies on the existence of an n-th cross effect functor, cr_nF, which measures the failure of F to be degree n. For a functor $F : \mathcal{C} \to \mathcal{D}G$ the second cross effect $cr_2F(X,Y)$ is the total homotopy fiber of the square diagram

$$\begin{array}{ccc} F(X\coprod_A Y) & \longrightarrow & F(X\coprod_A B) \\ \downarrow & & \downarrow \\ F(B\coprod_A Y) & \longrightarrow & F(B\coprod_A B) \end{array}$$

where A is the initial object of $\mathcal{C}$, B is the terminal object, and $\coprod_A$ is the coproduct in $\mathcal{C}$. A major difficulty in establishing this theory was verifying that the functor $\perp_n F(X) = cr_n F(X, \cdots, X)$ is part of a cotriple. This property is required to define $P_n F$. We obtained the following result as a concrete calculation using explicit models for homotopy pullbacks.

Theorem 1. *Let $\mathcal{C}$ be a model category. For a homotopy functor $F : \mathcal{C} \to \mathcal{D}G$, the functor $\perp_n F$ is part of a cotriple.*

In particular, we showed that the total homotopy fiber functor is part of a cotriple.

Our second accomplishment was to determine the values of the cross effect functors, essential building blocks for the terms in discrete Taylor towers, for a large class of examples in the unbased setting. For the category $\mathcal{D}GA^\eta$ of differential graded algebras factoring a fixed map $\eta : A \to B$, let $J(X)$ be the homotopy fiber of $U(X) \to U(B)$, where U is the forgetful functor to $\mathcal{C}h(k)$. Then our calculations specialize as follows.

Theorem 2. *For cofibrant X in $\mathcal{D}GA^\eta$, $cr_n J(X, \cdots, X) \simeq J(X)^{\otimes_k n}$ where $\otimes$ is the derived tensor product.*

Underlying these first two directions was the desire to work toward understanding André-Quillen homology as an example of calculus in the unbased setting, as the Kantorovitz-McCarthy results involved "adding a basepoint" in order to use the based calculus. In the last direction of our project, we set up a framework for relating the based calculus interpretation of André-Quillen homology to an unbased one, by relating the unbased derivative of J to the based derivative of J using an unpublished theorem of Randy McCarthy. Adapting this theorem to our setting will be a future direction of inquiry for our team.

7 G-categories and G-spaces: Anna Marie Bohmann, Kristen Mazur, Angelica Osorno*, Viktoriya Ozornova, Kate Ponto*, Caroline Yarnell

In [20], Thomason proved that the category of small categories admits a closed model structure. Additionally, he showed this structure is Quillen equivalent to the usual model structure on the category of simplicial sets. This equivalence implies that Cat is Quillen equivalent to the category of topological spaces with the standard model structure. An important implication of this theorem is that every homotopy type is represented by the classifying space of a category. We have shown that a similar result holds equivariantly.

Theorem 3. *If G is a finite group, there is a model structure on the category $GCat$ of small G-categories, and this category is Quillen equivalent to the standard model structure on the categroy of G-spaces.*

For any category $\mathcal{C}$ we can define a category $G\mathcal{C}$ of G-objects in $\mathcal{C}$ and a presheaf category $\mathcal{O}_G\mathcal{C}$. Recall that $\mathcal{O}_G$ is the category with objects G-sets G/H for H a subgroup of G and morphisms G-maps. The category of functors

$$X\colon \mathcal{O}_G^{op} \to \mathcal{C},$$

presheaves on $\mathcal{C}$, is denoted $\mathcal{O}_G\mathcal{C}$. There is a canonical functor

$$\Phi\colon G\mathcal{C} \to \mathcal{O}_G\mathcal{C}$$

that sends an object Y of $G\mathcal{C}$ to its system of fixed points, i.e, $\Phi(Y)(G/H) = Y^H$. This functor has a left adjoint Λ, which is given by $\Lambda(X) = X(G/e)$

Marc Stephan [19] has proved a general result that provides conditions on the category $\mathcal{C}$ that allow one to lift the projective model structure on $\mathcal{O}_G\mathcal{C}$ to a model structure on $G\mathcal{C}$, and further imply the fixed point functor Φ is a Quillen equivalence. As a result, much of the work in our project was verifying that conditions of his result are satisfied when $\mathcal{C}$ is the category of categories. In this case his conditions reduce to two conditions on the fixed point functors $(-)^H : G\mathcal{C} \to \mathcal{C}$. These functors must preserve some directed colimits and some pushouts.

More explicitly, we used Stephan's results [19] and some new results of our own to show that all the categories in the following diagram have model structures, and all the arrows of the diagram form Quillen equivalences:

$$\begin{array}{ccccc}
GCat & \underset{cSd^2}{\overset{Ex^2N}{\rightleftarrows}} & GsSets & \underset{|-|}{\overset{S_\bullet(-)}{\leftrightarrows}} & GTop \\
\Lambda\uparrow\ \downarrow\Phi & & \Lambda\uparrow\ \downarrow\Phi & & \Lambda\uparrow\ \downarrow\Phi \\
\mathcal{O}_G Cat & \underset{cSd^2}{\overset{Ex^2N}{\rightleftarrows}} & \mathcal{O}_G sSets & \underset{|-|}{\overset{S_\bullet(-)}{\leftrightarrows}} & \mathcal{O}_G Top.
\end{array}$$

8 Results of the Model Categories Team: Marzieh Bayeh, Kathryn Hess*, Varvara Karpova, Magdalena Kedziorek, Emily Riehl, Brooke Shipley*

8.1 The problem

Let $(\mathcal{M}, \mathtt{Fib}, \mathtt{Cof}, \mathtt{WE})$ be a model category, and $\mathcal{C}$ a bicomplete category. Given a pair of adjoint functors

$$L : \mathcal{M} \rightleftarrows \mathcal{C} : R,$$

there are well known conditions under which there is a model category structure on $\mathcal{C}$ with $R^{-1}(\mathtt{WE})$, $R^{-1}(\mathtt{Fib})$ as weak equivalences and fibrations, respectively. Our team studied the dual situation, where one has a pair of adjoint functors

$$L : \mathcal{C} \rightleftarrows \mathcal{M} : R \tag{1}$$

and wants to know when is there a model category structure on $\mathcal{C}$ with $L^{-1}(\mathtt{WE})$, $L^{-1}(\mathtt{Cof})$ as weak equivalences and cofibrations, respectively. We call this a *left-induced* model category structure.

One possible answer to this question can be formulated in terms of the following constructions. Let $\mathtt{X}$ be a class of morphisms in a complete category $\mathcal{C}$. Let $Y : \lambda^{op} \to \mathcal{C}$ be a functor, where λ is an ordinal. If for all $\beta < \lambda$, there is a pullback

$$\begin{array}{ccc}
Y_{\beta+1} & \longrightarrow & X'_{\beta+1} \\
\downarrow & & \downarrow{\scriptstyle x_{\beta+1}\in \mathtt{X}} \\
Y_\beta & \xrightarrow{k_\beta \in \mathcal{C}} & X_{\beta+1}
\end{array}$$

and $Y_\gamma := \lim_{\beta<\gamma} Y_\beta$ for all limit ordinals $\gamma < \lambda$, then the composition of the tower

$$\lim_{\lambda^{op}} Y_\beta \to Y_0,$$

is an $\mathtt{X}$-*Postnikov tower*. The class of all $\mathtt{X}$-Postnikov towers is denoted $\mathsf{Post}_{\mathtt{X}}$.

A *Postnikov presentation* of a model category $(\mathcal{M}, \mathtt{Fib}, \mathtt{Cof}, \mathtt{WE})$ consists of a pair of classes of morphisms $(\mathtt{X}, \mathtt{Z})$ satisfying

$$\mathtt{Fib} = \widehat{\mathsf{Post}_{\mathtt{X}}} \quad \text{and} \quad \mathtt{Fib} \cap \mathtt{WE} = \widehat{\mathsf{Post}_{\mathtt{Z}}},$$

where $\widehat{(.)}$ denotes the closure under retracts of a class of morphisms, and such that for all $f \in \operatorname{mor} \mathcal{M}$, there exist

- $i \in \texttt{Cof}$ and $p \in \mathsf{Post}_{\texttt{Z}}$ such that $f = pi$, and
- $j \in \texttt{Cof} \cap \texttt{WE}$ and $q \in \mathsf{Post}_{\texttt{X}}$ such that $f = qj$.

As shown in earlier work by the team leaders, given an adjoint pair (1), if $L\big(\mathsf{Post}_{R(\texttt{Z})}\big) \subset \texttt{WE}$, and for all $f \in \operatorname{mor} \mathcal{C}$ there exist

- $i \in L^{-1}(\texttt{Cof})$ and $p \in \mathsf{Post}_{R(\texttt{Z})}$ such that $f = pi$, and
- $j \in L^{-1}(\texttt{Cof} \cap \texttt{WE})$ and $q \in \mathsf{Post}_{R(\texttt{X})}$ such that $f = qj$,

then $\mathcal{C}$ admits a left-induced model category structure with Postnikov presentaion $\big(R(\texttt{X}), R(\texttt{Z})\big)$. This existence result had already been applied successfully by the team leaders to prove existence of left-induced model category structure in concrete cases [9].

8.2 Our results

Building on recent work of Makkai and Rosicky [16], we established the following existence result for model category structure on a category of coalgebras $\mathcal{M}_K$ over a comonad K acting on a model category $\mathcal{M}$.

Theorem 4. *Let $\mathcal{M}$ be a combinatorial model category in which the cofibrations are exactly the monomorphisms, K is an accessible comonad on $\mathcal{M}$ that preserves monomorphisms, $\texttt{Fib} \cap \texttt{WE} = \widehat{\mathrm{Post}}_Z$, and $U_K(\mathrm{Post}_{KZ}) \subset \texttt{WE}$ where $U_K \colon \mathbf{M}_K \to \mathbf{M}$ is the forgetful functor.*

The category of K-coalgebras $\mathcal{M}_K$ has a left-induced model category structure such that the class of acyclic fibrations is $\widehat{\mathrm{Post}}_{KZ}$ and such that the forgetful/cofree adjunction $U_K \colon \mathcal{M}_K \rightleftarrows \mathcal{M} : F_K$ is a Quillen pair.

We also constructed calculationally explicit fibrant replacements for K-coalgebras, under certain fibrancy conditions in the underlying category.

Theorem 5. *Let $\mathcal{M}$ be a simplicial model category with Postnikov presentation $(\texttt{Fib}, \texttt{Fib} \cap \texttt{WE})$, and let K be a comonad on $\mathcal{M}$. Let (M, ρ) be a K-coalgebra such that the shifted K-cobar construction on M is Reedy fibrant in $\mathcal{M}$. Then there exists a factorization in $\mathcal{M}_K$*

$$\begin{array}{ccc} M & \xrightarrow{\quad\rho\quad} & KM, \\ & {\scriptstyle\sim}\searrow \quad \nearrow{\scriptstyle \in \mathrm{Post}_{K(\texttt{Fib})}} & \\ & \operatorname{Tot} \Omega^\bullet_M & \end{array}$$

where $\Omega^\bullet_M$ denotes the usual K-cobar construction on M.

We were particularly interested in properties and existence of left-induced model category structures on categories of diagrams in a given model category, with respect to the adjunction $\iota^*_{\mathbf{D}} \colon \mathcal{M}^{\mathbf{D}} \rightleftarrows \mathcal{M}\mathrm{R}an_{\iota_{\mathbf{D}}}$ induced by the inclusion $\iota_{\mathbf{D}} : \mathbf{D}_\delta \to \mathbf{D}$ of the discrete subcategory.

Theorem 6. *Let $\mathcal{M}$ be a model category, and let $\mathbf{D}$ be a small category. Suppose that $\mathcal{M}^{\mathbf{D}}$ admits the left-induced model category structure.*

1. *If $\mathbf{M}$ is left or right proper, then so is $\mathbf{M}^{\mathbf{D}}$.*
2. *If $\mathbf{M}$ is a $\mathcal{V}$-model category for some monoidal model category $\mathcal{V}$, then so is $\mathcal{M}^{\mathbf{D}}$.*
3. *If $\mathcal{M}$ has Postnikov presentation $(\texttt{X}, \texttt{Z})$, then $\mathcal{M}^{\mathbf{D}}$ has Postnikov presentation $(\texttt{X} \times \mathbf{D}, \texttt{Z} \times \mathbf{D})$.*

Theorem 7.

1. *When $\mathbf{D}$ is a Reedy category, then Reedy model structure on $\mathcal{M}^{\mathbf{D}}$ has a Postnikov presentation given explicitly by a pullback-cotensor involving boundaries of representables and the generating (acyclic) fibrations.*
2. *If $\mathcal{M}$ is a Grothendieck abelian category, then $\mathbf{Ch}(\mathcal{M})^{\mathbf{D}}$ has the left induced (injective) model structure with Postnikov presentation.*

Remark 1. *Applying Theorem 5 to the comonad $\iota_{\mathbf{D}}^*\mathrm{Ran}_{\iota_{\mathbf{D}}}$, one obtains a model for fibrant replacements in $\mathcal{M}^{\mathbf{D}}$ that should be useful for homotopy limit calculations (and indeed might be the same as the Bousfield-Kan formula).*

We studied properties of left-induced model category structures in general as well.

Theorem 8. *Let $L : \mathcal{C} \rightleftarrows \mathcal{M} : R$ be an adjoint pair, where $\mathcal{M}$ is a model category, and $\mathcal{C}$ admits the left-induced model category structure.*

1. *If $\mathcal{M}$ is left proper, then $\mathcal{C}$ is left proper.*
2. *If $\mathcal{M}$ is a a $\mathcal{V}$-model category for some monoidal model category $\mathcal{V}$, $L \dashv R$ is a $\mathcal{V}$-adjunction, and $\mathcal{C}$ is a tensored and cotensored $\mathcal{V}$-category, then $\mathcal{C}$ is a $\mathcal{V}$-model category.*

Finally we observed that the proofs of several of the steps in establishing the existence of right Bousfield localizations do not actually require a full model structure and would therefore apply immediately to the dual setting with a Postnikov presentation. We intend to continue working in this direction.

9 Orbi Mapping Spaces as Groupoids: Vesta Coufal, Faten Labassi, Dorette Pronk*, Carmen Rovi, Laura Scull*, Courtney Thatcher

Background Orbispaces can be modeled using equivalence classes of proper étale groupoids, i.e., topological groupoids

$$G_1 \times_{G_0} G_1 \xrightarrow{m} G_1 \xrightarrow{i} G_1 \underset{t}{\overset{s}{\underset{\longleftarrow u}{\rightrightarrows}}} G_0$$

such that G_0 and G_1 are locally compact Hausdorff spaces, source and target maps $s, t \colon G_1 \rightrightarrows G_0$ are local homeomorphisms, the diagonal $(s,t)\colon G_1 \to G_0 \times G_0$ is a proper map. We refer to such a groupoid as an orbigroupoid. Two orbigroupoids $\mathcal{G}$ and $\mathcal{H}$ can represent the same orbispace, so we say that they are Morita equivalent if and only if there is a third groupoid $\mathcal{K}$ with essential equivalences $\mathcal{G} \leftarrow \mathcal{K} \to \mathcal{H}$, where the essential equivalence used is an internalization (for topological spaces) of the notion of weak equivalence between categories. Then orbispaces are represented by Morita equivalence classes of orbigroupoids.

Essential equivalences between topological groupoids are not invertible in general. However, they satisfy the conditions to form a bicategory of fractions. As a consequence an orbimap $\mathcal{G} \to \mathcal{H}$ between two orbigroupoids is given by a pair $\mathcal{G} \xleftarrow{\upsilon} \mathcal{K} \xrightarrow{\varphi} \mathcal{H}$, where υ is an essential equivalence. A 2-cell between two such maps is an equivalence class of diagrams

$$\begin{array}{ccccc} & & \mathcal{K} & & \\ & \swarrow^{\upsilon} & \uparrow \nu_1 & \searrow^{\varphi} & \\ \mathcal{G} & \alpha_1 \Downarrow & \mathcal{L} & \alpha_2 \Downarrow & \mathcal{H} \\ & \nwarrow_{\upsilon'} & \downarrow \nu_2 & \nearrow_{\varphi'} & \\ & & \mathcal{K}' & & \end{array}$$

where υ, υ', ν_1 and ν_2 are essential equivalences. We write **OrbiSpaces** to denote the resulting bicategory of orbigroupoids, orbimaps, and 2-cells between them.

Our Project The 2-cells in **OrbiSpaces** are all invertible with respect to the vertical composition, so we obtain a groupoid **OrbiSpaces**$(\mathcal{G}, \mathcal{H})$ of orbimaps and 2-cells between any two two orbigroupoids $\mathcal{G}$ and $\mathcal{H}$. This mapping groupoid is the main object of study of this project. Our project is to show that this carries the structure of an orbigroupid, and hence we could make the maps between orbispaces into a mapping space.

To do this, we first show that this groupoid can be given a topology to form an orbigroupoid $\mathrm{OMap}(\mathcal{G}, \mathcal{H})$. We show that with this topology, the bicategory **OrbiSpaces** is Cartesian closed, that is, for any orbigroupoids $\mathcal{L}$, $\mathcal{G}$ and $\mathcal{H}$ there is an equivalence of categories **OrbiSpaces**$(\mathcal{L} \times \mathcal{G}, \mathcal{H}) \simeq$ **OrbiSpaces**$(\mathcal{L}, \mathrm{OMap}(\mathcal{G}, \mathcal{H}))$.

We also work out in detail what the orbispaces of the form $\mathrm{OMap}(*_G, \mathcal{H})$ look like, where $*_G$ is the orbigroupoid with space of objects $\{*\}$ and space of arrows the discrete group G (with composition induced by the multiplication of G). Lastly, we consider the question of the hom-groupoids, in the bicategory of fractions. We show that $\mathrm{OMap}(\mathcal{G}, \mathcal{H})$ is the pseudo colimit of a diagram of groupoids of the form $\mathrm{GMap}(\mathcal{K}, \mathcal{H})$, where $\mathcal{K}$ has an essential equivalence into $\mathcal{G}$, and $\mathrm{GMap}(\mathcal{K}, \mathcal{H})$ is the orbigroupoid of groupoid homomorphisms and groupoid natural transformations from $\mathcal{K}$ to $\mathcal{H}$. This pseudo colimit construction carries a topology and allows us to define a true orbispace of maps between any two orbispaces.

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