The Alcove Walk Model and Matrix Coefficients

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Matrix Coefficients

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For λ an antidominant weight, define the Whittaker coefficient

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To evaluate, we integrate over the *double Iwasawa cells*

$$C_{\lambda\mu} := U^- t^\lambda K \cap U^+ t^\mu K,$$

where each $ut^\lambda = u' t^\mu k \in U^+ AK$.

Matrix Coefficients

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using the Bruhat decomposition. Therefore, we get a stratification on the double Iwasawa cells

$$\begin{aligned} C_{\lambda\mu} &= U^- t^\lambda K \cap U^+ t^\mu K \\ &= \bigcup_{\substack{w, w' \in W \\ v \in \widetilde{W}}} U^- t^\lambda wI \cap IvI \cap U^+ t^\mu w' I. \end{aligned}$$

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We can then rewrite the Whittaker coefficient as follows:

$$W(t^\lambda) = \frac{1}{\mathrm{vol}(K)} \sum_{\substack{w, w' \in W \\ v \in \widetilde{W}}} \chi(t^\mu) \left(\int_{U^- t^\lambda w I \cap I v I \cap U^+ t^\mu w' I} \psi(u) du \right).$$

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Claim: *This integral is actually extremely computable!*

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Theorem (Parkinson-Ram-Schwer)

Orient the affine hyperplanes so that the positive side faces a point deep in the antidominant Weyl chamber. Then there is a bijection

$$\left\{\begin{matrix} \textit{positively folded labeled alcove walks} \\ \textit{of type } v \textit{ ending at } t^{\lambda}w \end{matrix}\right\} \longleftrightarrow U^{-}t^{\lambda}wI \cap IvI.$$

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Need to evaluate $\psi(u)$; the labelings track the unipotent parts.

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Recall that ψ is trivial on $\mathbb{F}_q[[t]]$, and so this path contributes 0 to $W(t^{(1,-1)})$. (We knew this already since λ not antidominant.)

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The cells $C_{\lambda\mu} \cong \mathbb{F}_q^{(\#\text{positive crossings})} \times (\mathbb{F}_q^{\times})^{(\#\text{positive folds})}$.

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- First find all walks indexing points in $U^{-}t^{\lambda}wI \cap IvI$.

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- Among those, see which ones also correspond to walks indexing points in $U^{+}t^{\mu}w'I \cap IvI$.

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- Read off the labelings of the walks from both steps to evaluate the character on the unipotent part.

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For $SL_2(\mathbb{F}_q((t)))$, we recover Tokuyama's formula ***bijectively****.*

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Theorem (B-Brubaker)

*For $SL_2(\mathbb{F}_q((t)))$, we recover Tokuyama's formula **bijectively**. Roughly speaking, each Gelfand-Tsetlin pattern corresponds to a stratum in $C_{\lambda\mu}$; the statistics are recording its weighted volume.*