Specialized Macdonald polynomials, quantum $K$-theory, and Kirillov-Reshetikhin crystals

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Whittaker Functions: Number Theory, Geometry, and Physics
Banff International Research Station, October 2013
Macdonald polynomials

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Recursive construction procedure (for the non-symmetric ones $E_\mu(x; q, t)$), based on Cherednik’s intertwiners $I_i$. 
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$$\hat{\Psi}_\lambda(x; q) := \Psi_\lambda(x; q) \prod_{i \in I} \prod_{r=1}^{\langle \lambda, \alpha_i^\vee \rangle} (1 - q^r).$$
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**Theorem (Braverman-Finkelberg, Ion)**

We have

$$P_\lambda(x; q, t = 0) = \hat{\Psi}_\lambda(x; q).$$
Schubert calculus

Flag variety $G/B$, Schubert variety $X_w = \overline{B^{-} w B}/B$, for $w \in W$.  

$H^\ast(G/B)$ and $K(G/B)$ have bases of Schubert classes; for $K$-theory, they are the classes $[O_w] = [O_{X_w}]$ of structure sheaves of $X_w$.

The quantum cohomology algebra $QH^\ast(G/B)$ still has the Schubert basis, but over $\mathbb{C}[q_1, \ldots, q_r]$. The structure constants (for multiplying Schubert classes) are the 3-point Gromov-Witten (GW) invariants. A $k$-point GW invariant (of degree $d$) counts curves of degree $d$ passing through $k$ given Schubert varieties.
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**Theorem (Braverman-Finkelberg)**

*In simply-laced types, the q-Whittaker function $\Psi_{\lambda}(x; q)$ (viewed as a function of $\lambda$) coincides with the $K$-theoretic $J$-function.*
Kirillov-Reshetikhin (KR) modules

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Let $p = (p_1, p_2, \ldots)$ be a composition, and

$$W^p = W^{p_1,1} \otimes W^{p_2,1} \otimes \ldots, \quad \lambda = \omega_{p_1} + \omega_{p_2} + \ldots.$$
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Main Theorem (L.-Naito-Sagaki-Schilling-Shimozono)

For all untwisted affine root systems \(A^{(1)}_{n-1} - G^{(1)}_2\), we have

\[P_\lambda(x; q, 0) = X_\lambda(x; q).\]
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For all untwisted affine root systems $A_{n-1}^{(1)} - G_2^{(1)}$, we have

$P_\lambda(x; q, 0) = X_\lambda(x; q)$.

Remarks. (1) The result is believed to extend to the twisted types.
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(2) In simply-laced types, certain affine Demazure characters were identified with $P_\lambda(x; q, 0)$ (Ion), and $X_\lambda(x; q)$ (Fourier-Littelmann).
The underlying combinatorics

The quantum alcove model (L. and Lubovsky) describes all the mentioned structures:
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The model is uniform for all Lie types $A_{n-1} - G_2$. 
Finite root systems $\Phi \subset \mathfrak{h}^*_R$

Reflections $s_\alpha$, $\alpha \in \Phi$. 
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Comes from the multiplication of Schubert classes in the quantum cohomology of flag varieties $QH^*(G/B)$ (Fulton and Woodward).
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The quantum Bruhat graph $\text{QBG}(W)$ on $W$ is the directed graph with labeled edges

$$w \xrightarrow{\alpha} ws_\alpha,$$

where

$$\ell(ws_\alpha) = \ell(w) + 1 \quad (\text{covers of the Bruhat order}),$$

or

$$\ell(ws_\alpha) = \ell(w) - 2\text{ht}(\alpha^\vee) + 1 \quad (\text{ht}(\alpha^\vee) = \langle \rho, \alpha^\vee \rangle).$$
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Bruhat graph for $S_3$:
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\begin{align*}
321 & \rightarrow 312 & \alpha_{12} & \rightarrow & \alpha_{13} & \rightarrow & 231 & \rightarrow & \alpha_{23} & \rightarrow & 213 & \rightarrow & \alpha_{13} & \rightarrow & 132 \\
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The quantum alcove model

Given a dominant weight $\lambda$, we associate with it a sequence of roots, called a $\lambda$-chain:

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**Fact.** The construction of a $\lambda$-chain is based on a reduced decomposition of the affine Weyl group element corresponding to $A_\circ - \lambda$. This gives a sequence of alcoves from $A_\circ$ to $A_\circ - \lambda$. 
The quantum alcove model (cont.)

Given $\Gamma = (\beta_1, \ldots, \beta_m)$, let $r_i := s_{\beta_i}$. 
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$$J = (j_1 < \ldots < j_s) \subseteq \{1, \ldots, m\}.$$
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For $w \in W$ and $J$, construct the chain $\pi(w, J)$ of elements in $W$:

$$w_0 = w, \ldots, w_i := wr_{j_1} \ldots r_{j_i}, \ldots, w_s = \text{end}(w, J).$$
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Important structures:

$$\mathcal{A}_q(\Gamma, w) := \{J : \pi(w, J) \text{ path in } \text{QBG}(W)\},$$
$$\mathcal{A}_<(\Gamma, w) := \{J : \pi(w, J) \text{ saturated chain in } (W, <)\}.$$
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Let $\mathcal{A}_q(\Gamma) := \mathcal{A}_q(\Gamma, 1_W)$ and $\mathcal{A}_< (\Gamma) := \mathcal{A}_<(\Gamma, 1_W)$.  

Macdonald polynomials: the Ram-Yip formula

Given a dominant weight $\lambda$, consider a $\lambda$-chain $\Gamma := (\beta_1, \ldots, \beta_m)$. 

Theorem (Ram-Yip, L.)

$$P_\lambda(X; q, 0) = \sum_{J \in A_q(\Gamma)} q^{\text{height}(J)} x^{\text{weight}(J)}.$$ 

Remark. For $q = 0$, we retrieve the alcove model (L. and Postnikov, cf. Gaussent and Littelmann, Littelmann):

$$P_\lambda(X; 0, 0) = \chi(V_\lambda) = \sum_{J \in A_\nless(\Gamma)} x^{\text{weight}(J)}.$$
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$K(G/B)$ and $QK(G/B)$: Chevalley formulas

Recall: $K(G/B)$ and $QK(G/B)$ have bases of Schubert classes $[\mathcal{O}_{\chi_w}] = [\mathcal{O}_w], \ w \in W$. 

Theorem (L.-Postnikov, L.-Shimozono)

In $K(G/B)$ (finite-type or Kac-Moody), we have

$[\mathcal{O}_w] \cdot [\mathcal{O}_{s_k}] = \sum_{J \in A \prec (\Gamma_{\text{rev}}, w)} \{\emptyset\} (-1)^{|J|-1}[\mathcal{O}_{\text{end}}(w, J)]$.

Conjecture (L.-Postnikov)

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Remark. Restricting the RHS, we retrieve the Chevalley formula in $QH^*(G/B)$ (Fulton-Woodward).
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Recall: \( K(G/B) \) and \( QK(G/B) \) have bases of Schubert classes \([\mathcal{O}_{\chi_w}] = [\mathcal{O}_w], \ w \in W.\)

Let \( \Gamma_{\text{rev}} \) = reverse of an \( \omega_k \)-chain (\( \omega_k \) a fundamental weight).
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**Conjecture (L.-Postnikov)**

In $QK(G/B)$ (finite-type), we have:

\[ [O_w] \ast [O_{s_k}] = \sum_{J \in A_q(\Gamma_{\text{rev}}, w) \setminus \{\emptyset\}} (-1)^{|J| - 1} q_1^* \cdots q_r^* [O_{\text{end}(w, J)}]. \]
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$$[O_w] \ast [O_{s_k}] = \sum_{J \in A_q(\Gamma_{rev}, w) \setminus \{\emptyset\}} \left( -1 \right)^{|J| - 1} q_1^{*} \cdots q_{r}^{*} [O_{\text{end}(w, J)}].$$

**Remark.** Restricting the RHS, we retrieve the Chevalley formula in $QH^*(G/B)$ (Fulton-Woodward).
Evidence for the conjectured formula in $QK(G/B)$

- Computer experiments (A. Buch).

• Based on some relations in $QK(SL_n/B)$ discovered by Kirillov-Maeno, we constructed polynomials $G_w(x; q)$, called quantum Grothendieck polynomials.

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Kirillov-Reshetikhin (KR) modules/crystals

Recall the KR modules, as modules for $U_q(\hat{g})$: $W^{r,s}$ and

$$W \otimes^p = W^{p_1,1} \otimes W^{p_2,1} \otimes \ldots .$$
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Kashiwara (crystal) operators are modified versions of the Chevalley generators (indexed by the simple roots): $\tilde{f}_0, \ldots, \tilde{f}_r$. 

Fact. The crystal structure on $B \otimes p$ is defined by a tensor product rule:

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$$\tilde{f}_i : B \to B \sqcup \{0\}, \quad \tilde{f}_i b = b' \iff b \overset{i}{\longrightarrow} b'.$$
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Models for KR crystals

Fact. In the classical types $A - D$ there are tableau models (the usual column-strict fillings in type $A_{n-1}^{(1)}$, but more involved in the other types, particularly for $B_n^{(1)}$ and $D_n^{(1)}$).
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Goal. Uniform model for all types $A_{n-1}^{(1)} - G_2^{(1)}$, based on the quantum alcove model.
The quantum alcove model for KR crystals

Given \( \mathbf{p} = (p_1, p_2, \ldots) \) and an arbitrary Lie type, let

\[
\lambda = \omega_{p_1} + \omega_{p_2} + \ldots.
\]

Construction. (L. and Lubovsky, generalization of L.-Postnikov, Gaussent-Littelmann)

Crystal operators \( \tilde{f}_1, \ldots, \tilde{f}_r \) and \( \tilde{f}_0 \) on \( A_q(\Gamma) \).

Main Theorem (L.-Naito-Sagaki-Schilling-Shimozono)

The (combinatorial) crystal \( A_q(\Gamma) \) is isomorphic to the tensor product of KR crystals \( B \otimes \mathbf{p} \).
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The (combinatorial) crystal $A_q(\Gamma)$ is isomorphic to the tensor product of KR crystals $B^\otimes p$. 
The energy function

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More precisely, $D_B : B \to \mathbb{Z}_{\geq 0}$ satisfies the following conditions:

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Goal. A more efficient uniform calculation, based only on the combinatorial data associated with a crystal vertex (type $A$: Lascoux–Schützenberger charge statistic).
The energy via the quantum alcove model

Consider \( J = \{j_1 < j_2 < \ldots < j_s\} \) in \( \mathcal{A}_q(\Gamma) \) for \( \Gamma = (\beta_1, \ldots, \beta_m) \), i.e., we have a path in the quantum Bruhat graph

\[
1_W = w_0 \xrightarrow{\beta_{j_1}} w_1 \xrightarrow{\beta_{j_2}} \cdots \xrightarrow{\beta_{j_s}} w_s.
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Recall that $\text{height}(J)$ measures the down steps in the above path.
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Recall that \( \text{height}(J) \) measures the down steps in the above path.

**Theorem (L.-Naito-Sagaki-Schilling-Shimozono)**

*Given \( J \in A_q(\Gamma) \), which is identified with \( B^\otimes p \), we have*

\[
D_B(J) = -\text{height}(J) .
\]
The combinatorial $R$-matrix via the quantum alcove model

This is the (unique) affine crystal isomorphism which commutes factors in the tensor product of KR crystals $B \otimes p$
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**Theorem (L.-Lubovsky)**

*We give a uniform realization, based on the quantum alcove model, of the combinatorial $R$-matrix.*
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**Theorem (L.-Lubovsky)**

*We give a uniform realization, based on the quantum alcove model, of the combinatorial $R$-matrix.*

We use combinatorial moves based on certain operators on $W$ defined by $\text{QBG}(W)$, which satisfy the Yang-Baxter equation (Brenti-Fomin-Postnikov).
Example in type $A_2$.

\[ \mathbf{p} = (1, 2, 2, 1) = \begin{array}{cccc} & & & \\
& & & \\
& & & \\
& & & \\
\end{array} \;; \quad \lambda = \omega_1 + \omega_2 + \omega_2 + \omega_1 = (4, 2, 0). \]
Example in type $A_2$.

$$\mathbf{p} = (1, 2, 2, 1) = \begin{array}{cccc}
\downarrow & \downarrow & \downarrow & \downarrow \\
\downarrow & \downarrow & \downarrow & \downarrow \\
\downarrow & \downarrow & \downarrow & \downarrow \\
\downarrow & \downarrow & \downarrow & \downarrow \\
\end{array}; \quad \lambda = \omega_1 + \omega_2 + \omega_2 + \omega_1 = (4, 2, 0).$$

A $\lambda$-chain as a concatenation of $\omega_1$-, $\omega_2$-, $\omega_2$-, and $\omega_1$-chains:

$$\Gamma = ( (1, 2), (1, 3) | (2, 3), (1, 3) | (2, 3), (1, 3) | (1, 2), (1, 3) ).$$
Example. Let $J = \{1, 2, 3, 6, 7, 8\}$.

\[
\left( (1,2), (1,3) | (2,3), (1,3) | (2,3), (1,3) | (1,2), (1,3) \right).
\]
Example. Let $J = \{1, 2, 3, 6, 7, 8\}$.

\[
( (1, 2), (1, 3) \mid (2, 3), (1, 3) \mid (2, 3), (1, 3) \mid (1, 2), (1, 3) ) .
\]

Claim: $J$ is admissible. Indeed, the corresponding path in the quantum Bruhat graph is

\[
\begin{align*}
&1 \quad 2 \quad 3 \\
&2 \quad 1 \quad 3 \\
&3 \quad 2 \quad 1 \\
&1 \quad 2 \quad 3
\end{align*}
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\]

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\[
\begin{array}{ccccccc}
& 1 & > & 2 & < & 3 & > \\
2 & \downarrow & 1 & \downarrow & 1 & \downarrow & 3 \\
3 & 1 & 2 & 2 & 1 & 3 & 2
\end{array}
\]

The corresponding element in $B^\otimes p = B^{1,1} \otimes B^{2,1} \otimes B^{2,1} \otimes B^{1,1}$ represented via column-strict fillings:

\[
\begin{array}{ccc}
3 & \otimes & 2 \\
3 & \otimes & 1 \\
2 & \otimes & 3
\end{array}
\]
The energy calculation

**Example.** Consider the running example: $\lambda = \omega_1 + \omega_2 + \omega_2 + \omega_1$ in type $A_2$. 

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Example. Consider the running example: \( \lambda = \omega_1 + \omega_2 + \omega_2 + \omega_1 \) in type \( A_2 \).
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\]

\[
(h_i) = ( 2, 4 \mid 2, 3 \mid 1, 2 \mid 1, 1 ).
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We considered the $\lambda$-chain $\Gamma$ and $J = \{1, 2, 3, 6, 7, 8\} \in \mathcal{A}(\Gamma)$:

$$\Gamma = \begin{pmatrix} (1, 2), (1, 3) | (2, 3), (1, 3) | (2, 3), \underline{(1, 3)} | (1, 2), (1, 3) \end{pmatrix},$$

$$\left( h_i \right) = \begin{pmatrix} 2, & 4 | 2, & 3 | 1, & 2 | 1, \end{pmatrix}.$$

We have

$$\text{height}(J) = 2.$$