Whittaker Functions: Number Theory, Geometry and Physics 13w5154

Ben Brubaker (Univ. of Minnesota),
Daniel Bump (Stanford University),
Gautam Chinta (City College – New York)
Solomon Friedberg (Boston College)
Paul E. Gunnells (UMass – Amherst)

10/13/2013–10/18/2013

1 An Overview of the Workshop

What are now understood as $GL(2, \mathbb{R})$ Whittaker functions were initially defined by Whittaker as solutions to the confluent hypergeometric differential equation [19]. H. Jacquet used the term "Whittaker function" in his thesis to refer to a more general class of special functions on a reductive group G over a local field F [13]. In the case of $G = SL(2)$ and $F = \mathbb{R}$, this reduces to the original definition.

As we discussed during the workshop, and will explain in the following pages, Whittaker functions exhibit what one might call an "unreasonable eclecticness" as they seem to arise in critical roles in many different contexts in mathematics and physics. At present, many of these connections remain relatively unexplored. The goal of the workshop was to report on and chart future progress toward such connections.

Here are several prominent appearances of Whittaker functions, in areas where there has been recent important activity. We observe that these connections tend to point in the direction of deeper connections with quantum groups and geometry (including geometric Langlands theory).

- In number theory, Whittaker functions appear as the Fourier coefficients of automorphic forms, used to define their associated L-functions (cf. [17]). In particular, when applied to Eisenstein series, they arise in both the Langlands-Shahidi method and in Rankin-Selberg constructions. Recently there has been a push to generalize this theory to infinite-dimensional groups with applications to number theory (cf. [5]). Moreover Eisenstein series are appearing in string theory [11].

- Stade studied integrals of Whittaker functions motivated by number theory [18]. These have resurfaced in several of the connections noted below including those from physics.

- In geometry, Whittaker functions reflect the geometry of the flag variety of the Langland dual group. Connections with Kazhdan-Lusztig theory and the equivariant cohomology (and K-theory) of Schubert and Bott-Samelson varieties can be seen [3].

- In quantum physics, archimedean Whittaker functions for real groups were shown to be common eigenfunctions of the quantum Toda Hamiltonians, leading to the total integrability of the Toda lattice (Kostant [15], Kazhdan) and the mirror conjecture for flag manifolds (Givental–Lee [10]).

- In statistical physics, nonarchimedean Whittaker functions may be expressed as partition functions for the six-vertex model (cf. [1]), which was the crucial example leading to the invention of quantum groups by Drinfeld and Jimbo (cf. [6]).

- In combinatorics, Whittaker functions may be expressed in terms of Macdonald polynomials [7]. For nonarchimedean Whittaker functions, Demazure-Lusztig operators relate different Whittaker functions [3]. From this point of view, the connections with geometry (mentioned above) can be seen very clearly.

- In different cases Whittaker functions may be described as sums over Kashiwara crystals, which are combinatorial analogs of Lie group representations [14]. Crystals are "crystalized" representations of quantum groups.

- In the archimedean case, Gerasimov, Lebedev and Oblezin [9] on the one hand and Ishii and Oda [12] on the other hand obtained results that may be understood in terms of crystals. The Whittaker function is expressed as a sum over Gelfand-Tsetlin patterns, which are in bijection with a suitable crystal (cf. [16]).

- In the theory of Brownian motion, beginning with work of Biane, Bougerol and O'Connell [2] a deep connection was found with the Littelmann Path Method [16] and the theory of crystals. Whittaker functions occur naturally in this theory. After further work of O'Connell and his collaborators this led to Chhaibi's dissertation relating Whittaker functions to "geometric" crystals, which are schemes with structures whose tropicalizations are Kashiwara crystals [8].

- The above connections with crystals relate to archimedean Whittaker functions. It was found by the organizers that nonarchimedean (p-adic) Whittaker functions may also be expressed as sums over Kashiwara crystals (cf. [4]). This includes Whittaker functions on metaplectic covering groups.

These themes were among the ones represented at this workshop. They give an indication of the breadth and multiple connections of this topic.

2 A Brief Introduction to Whittaker Functions

Let G be a split, reductive group over a local field F with maximal torus T and unipotent subgroup U corresponding to a choice of positive roots. Consider functions f on $G(F)$

satisfying the transformation property:

$$f(ug) = \psi(u)f(g) \qquad \text{where } \psi \text{ is a non-degenerate character of } U.$$

Then $G(F)$ acts on such functions by right translation. Any function appearing in an irreducible subspace of this regular representation is called a *Whittaker function*. A famous result, attributable in various cases to Gelfand-Graev, Jacquet, Langlands, Piatetski-Shapiro, Shalika, and Rodier, states that an irreducible representation (π, V) of $G(F)$ has at most one space of Whittaker functions isomorphic to π as a representation of G.

A particularly rich class of representations over a local field are the so-called *unramified principal series.* Begin with a character χ of $T(F)$ trivial on $T(F) \cap K$, where K denotes a maximal compact subgroup of G. Then form the induced representation $(\pi, V_\chi) = \text{Ind}_B^G(\chi)$ by considering χ as a representation of $B = TU$, a Borel subgroup. Thus

$$V_\chi = \left\{ f \colon G(F) \longrightarrow \mathbb{C} \mid f(bg) = \delta^{1/2}\chi(b)f(g) \right\}$$

where the f are assumed to be smooth. The action of G on this space of functions is again by right translation. For a generic choice of χ, the corresponding principle series representation is irreducible and has a Whittaker model.

Given f in V_χ, the corresponding Whittaker function may be constructed via the integral

$$W_f(g) = \int_U f(w_0 ug)\psi(u)^{-1}\, du.$$

Depending on the choice of χ, the integral may not converge. In this case, the integral must be suitably interpreted.

One very natural choice for f is the spherical function, denoted f°. It is the unique-up-to-constant K-fixed vector in the unramified principal series. Explicitly, writing any $g \in G$ as $g = bk$ with $b \in B, k \in K$, then

$$f^\circ(g) = f^\circ(bk) = \delta^{1/2}\chi(b).$$

If F is a non-archimedean local field, there is a beautiful formula for the values of the resulting spherical Whittaker function W_{f° at torus elements in $T(F)/T(F) \cap K$. In increasing levels of generality, it is due to Shintani, Kato, and Casselman-Shalika, and we refer to it in brief as the Casselman-Shalika formula. To state it, first recall that the unramified characters χ of $T(F)$ correspond to elements $\boldsymbol{z}$ in the dual torus $\hat{T}(\mathbb{C})$. Then the spherical Whittaker function at fixed element $a_\lambda \in T(F)/T(F) \cap K$ for a dominant weight λ, viewed as a function of $\boldsymbol{z}$ is given by

$$W_{f^\circ}(a_\lambda) = \prod_{\alpha \in \Phi^+} (1 - q^{-1}\boldsymbol{z}^\alpha) s_\lambda(\boldsymbol{z}),$$

where the product is taken over all positive roots and q is the cardinality of the residue field of F. Moreover, s_λ denotes the character of the *finite dimensional* irreducible representation of the dual group $\hat{G}(\mathbb{C})$ with highest weight λ. For example, if $G = GL_r(F)$ then $\hat{G} = GL_r(\mathbb{C})$ and s_λ is just the Schur polynomial corresponding to the partition λ.

This highest weight character can, in turn, be expressed in several ways. One might use the Weyl character formula, averaging an alternating sum over the Weyl group of G. Alternately, one could note that by the Borel-Weil theorem, for λ dominant, the space of global sections $\mathrm{H}^0(\hat{G}/\hat{B}, \mathcal{L}_\lambda) \simeq V_\lambda$ where $\mathcal{L}_\lambda$ is the line bundle on the complex flag variety $\hat{G}/\hat{B}$ corresponding to λ. Finally, one could use a combinatorial realization of the character as a generating function over a set in bijection with basis vectors for the highest weight representation (e.g. tableaux or Gelfand-Tsetlin patterns). These three ways of looking at a highest weight character correspond to connections with other fields noted in the introduction.[1]

3 Presentation Highlights

The workshop began with two expository talks by organizers, designed to give participants—including graduate students and junior researchers—an overview of the workshop themes. (Such presentations had been requested by participants in a pre-conference survey undertaken by the organizers). This was followed by talks highlighting recent developments and open problems concerning Whittaker functions in number theory, geometry, combinatorics, and physics. Most talks were videotaped. In the following we summarize the highlights of each talk. Junior speakers are indicated by a bullet (•).

Elizabeth Beazley (•) (Haverford College) spoke on *The alcove path model and matrix coefficients*. Formulas for spherical and Whittaker functions as sums over crystals may be obtained by reformulating Tokuyama's formula. The crystals can also be realized as (generalized) Mirkovic-Vilonen cycles, which are naturally indexed by the positively folded alcove walks originally defined by Gaussent-Littelmann and generalized by Parkinson-Ram-Schwer. So it is natural to compute matrix coefficients for spherical functions using the combinatorics of alcove walks. This lecture concerned the question of how to do so, that is, how to the use of the alcove path model to compute the matrix coefficients for spherical functions. The work presented is joint work with Benjamin Brubaker.

Daniel Bump (Stanford University) spoke on *Whittaker functions and quantum groups*. The goal of his talk was to explain the connections between these topics and to provide additional context on related research. The talk began with an exposition of work by Brubaker, Bump, Friedberg, Chinta and Gunnells on the representation of Whittaker functions by solvable lattice models. This was followed by a discussion of the how the Yang-Baxter equation was used by Drinfeld and by Faddeev, Reshetikhin and Takhtajan to construct (dual) quasitriangular Hopf algebras, or quantum groups. Variants of this construction by Cotta-Ramusino, Lambe and Rinaldi and by Buciumas which are applicable to the case in question were also discussed.

Reda Chhaibi (•) (Universität Zürich) spoke on *Archimedean Whittaker functions and geometric crystals*. This lecture focussed on the interplay between the representations of Lie groups and probability theory. He described a new path model for geometric crystals in the sense of Berenstein and Kazhdan, for complex semi-simple Lie groups, and the role of the theory of total positivity. This was used to explain new connections between Whittaker

[1] For the last of these, the usual generating function admits a deformation supported on a set of tableaux or patterns in bijection with certain statistical mechanical models.

functions and geometric crystals. This connection makes use of probability models and in particular Brownian motion, which is used to find the canonical measure on geometric crystals. This gives a new approach to the archimedean Whittaker integrals of Gerasimov, Lebedev, and Oblezin and suggests a way to unify their treatment with the non-archimedean case.

Dan Ciubotaru (University of Utah) spoke on *Classification of generic unitary irreducible representations*. He explained joint work with Dan Barbasch on a classification of all generic (Whittaker) unitary irreducible representations with Iwahori fixed vectors for a quasi-split simple p-adic group.

Solomon Friedberg (Boston College) spoke on *Metaplectic Whittaker fuctions*. He reported on the connection between the Whittaker coefficients of Eisenstein series on covers of groups and crystal graphs. He illustrated this with an account of covers of $GL(2)$, explaining how the Fourier coefficients of the Kubota Eisenstein series could be understood using quantum SL_2, and indicated generalizations, including his recent work with Lei Zhang on covers of odd orthogonal and symplectic groups. Even for the 1-fold cover, this work gives a new deformation of the Weyl character formula analogous to Tokuyama's result but not previously known. He also briefly described McNamara's work on local Whittaker coefficients for covers of the general linear group and their relation to Mirkovic-Vilonen cycles. Generalizations would be of great interest.

Holley Friedlander (•) (Williams College) presented *On the formulas of Tokuyama and Gindikin - Karpelevich for G_2*. She presented a conjecture, supported by a great deal of computer experiment, giving a formula analogous to the Tokuyama formula for the root system G_2 via Berenstein-Zelevinksy-Littelmann patterns. This is joint work with Paul Gunnells. As evidence for their conjecture, she explained that the formula exhibits the Gindikin- Karpelevich integral as a sum over the crystal $B(\infty)$, in the spirit of the papers of Bump-Nakasuji and McNamara.

P. Edward Herman (•) (University of Chicago) presented *On Patterson's Conjecture: Sums of Quartic Exponential Sums*. He presented a result that gives evidence for Patterson's conjecture on sums of exponential sums, by getting an asymptotic for a sum of quartic exponential sums over $\mathbb{Q}[i]$. Previously, the strongest evidence of Patterson's conjecture over a number field is the paper of Livné and Patterson on sums of cubic exponential sums over $\mathbb{Q}[\omega], \omega^3 = 1$. The key ideas in his result are a Kuznetsov-like trace formula for metaplectic forms over a quartic cover of GL_2 and an identity on exponential sums relating Kloosterman sums and quartic exponential sums. An unexpected aspect of the asymptotic of the sums of exponential sums is that there can be a secondary main term additional to the main term, something which is not predicted in Patterson's original paper.

Yumiko Hironaka (Waseda University) spoke on *Spherical functions on certain p-adic homogeneous spaces*. She introduced a spherical function on certain p-adic homogeneous spaces (weak spherical homogeneous satisfying some technical conditions), and gave a formula for it by using spherical functions of groups and functional equations. Then she discussed some specific spaces and gave explicit formulas by using specialized Macdonald polynomials associated to a root system. For the space of unramified hermitian matrices or unitary hermitian matrices, this was used to give a parametrization of all the spherical functions and a Plancherel formula on the Schwartz space.

Axel Kleinschmidt (•) (Max-Planck-Institut für Gravitationsphysik) spoke on *Whittaker vectors and constant terms for Kac-Moody groups*. He explained the link between Whittaker coefficients and quantum gravity. An important question in quantum gravity is to compute the correction to supergravity, which gives an effective model for massless states, by using string theory. This question turns out to require the study of Eisenstein series induced from the Borel subgroup attached to various E_n. Computing the constant term of such a series as a sum of intertwining operators has been carried out and shown to give the perturbative term. Higher order, non-perturbative, terms should correspond to integrals of Eisenstein series over other unipotent subgroups.

Kyu-Hwan Lee (University of Connecticut) spoke on *Eisenstein series on rank 2 hyperbolic Kac-Moody groups* (joint work with Lisa Carbone and Dongwen Liu). He defined an Eisenstein series on rank 2 hyperbolic Kac-Moody groups over the reals induced from a quasi-character. He explained the convergence of the constant term and hence the almost everywhere convergence of the Eisenstein series. He then defined and calculated the degenerate Fourier coefficients. He also considered Eisenstein series induced from cusp forms and sketched the proof that these are entire functions.

Cristian Lenart (State University of New York at Albany and Max-Planck-Institut für Mathematik, Bonn) spoke on *Specialized Macdonald polynomials, quantum K-theory, and Kirillov-Reshetikhin crystals*. Braverman and Finkelberg related the specialized symmetric Macdonald polynomials $P_\lambda(x;q,t=0)$ to the quantum K-theory $QK(G/B)$ of flag varieties (more precisely, to the K-theoretic J-functions) via their q-Whittaker functions. In this talk, the combinatorics underlying this connection and its ramifications were discussed. On the one hand, by the Ram-Yip formula, $P_\lambda(x;q,0)$ is expressed in terms of the so-called quantum alcove model. On the other hand, Lenart and Postnikov conjectured that the Chevalley multiplication formula in $QK(G/B)$ is expressed in terms of the same model; some evidence was discussed. Furthermore, the speaker explained that the quantum alcove model is also a uniform model, in all untwisted affine types, for tensor products of one-column Kirillov-Reshetikhin crystals (joint with Naito, Sagaki, Schilling, and Shimozono).

Maki Nakasuji (Sophia University in Tokyo) spoke on *Casselman's basis and Schubert calculus from computational evidence*. The Gindikin-Karpelevich formula is an important tool in the theory of automorphic forms. It calculates the effect of intertwining integrals on the spherical vector in a principal series representation. Bump and Nakasuji had conjectured a generalization of the Gindikin-Karpelevich formula for Iwahori fixed vectors that conjecturally takes a simple form depending on the smoothness of a corresponding Schubert cell in the flag variety of the Langland L-group. Nakasuji reported on recent progress by her and Naruse on this conjecture. They introduce new ideas to this problem, beginning with the introduction of the nil Hecke ring of Kostant and Kumar and a firmer connection with K-theory. It seems likely that this work will lead to a proof.

Daniel Orr (Virginia Tech) spoke on *Specializations of non-symmetric McDonald polynomials at infinity*. He presented joint work wtih Shimozono concerning alcove path formulas for specializations of nonsymmetric McDonald polynomials at $q=\infty$ and at $t=\infty$. At $q=\infty$, these polynomials were shown recently by Brubaker, Bump, and Licata to coincide with certain p-adic Iwahori-Whittaker functions, while at $t=\infty$ they are conjecturally related to the PBW filtration of level-one affine Demazure modules.

Manish Patnaik (•) (University of Alberta) spoke on *Eisenstein series on loop groups*. He explained aspects of the theory of Eisenstein series on loop groups and compared and contrasted them with Eisenstein series on Lie groups. In particular, he discussed aspects of his work with Garland and Miller in which they prove the entirety of loop group Eisenstein series induced from cusp forms on the underlying finite dimensional group, by demonstrating their absolute convergence on the full complex plane. This contrasts with finite-dimensional setting, where such series only converge absolutely in a right half plane (and have poles elsewhere coming from Langlands L-functions in their constant terms). This result is an analogue over the rationals of a theorem of Braverman and Kazhdan, who showed in the function field setting that the analogous Eisenstein series are finite sums. The exponentially rapid decrease of cusp forms as one approaches a cusp plays an important role in the proof.

Anna Puskás (•) (Columbia University) spoke on *Sums of metaplectic Demazure-Lusztig Operators and crystals*. A result of Brubaker, Bump and Licata concerns a natural basis of the Iwahori fixed vectors in the Whittaker model of an unramified principal series representation of a split semisimple p-adic group, indexed by the Weyl group. They show that the elements of this basis may be computed from one another by applying Demazure-Lusztig operators. The precise identities involve correction terms, which may be calculated by a combinatorial algorithm that is identical to the computation of the fibers of the Bott-Samelson resolution of a Schubert variety. This talk described the generalization of some of these ideas to the metaplectic context, and in particular introduced a new notion of metaplectic Demazure-Lustztig operators. It is joint work with Chinta and Gunnells.

Andre Reznikov (Bar-Ilan University) spoke on joint work with J. Bernstein on *Adelic action on periods of automorphic functions and special values of L-functions*. This talk concerned invariant functionals defined on automorphic representations via period integrals. (One example of such a functional is the Whittaker functional.) By considering the action of an adelic subgroup on such an invariant functional, the speaker showed that in certain cases this action gives rise to another period integral. In some cases, this corresponds to a known relation of an automorphic period to a special value of an appropriate L-function (e.g. the classical formulas of Hecke-Jacquet-Langlands and of Waldspurger). However, even in some of the simplest cases, he and Bernstein have found that the relation to L-functions is more puzzling, as it leads to a non-standard Euler product which nevertheless can be regularized by an appropriate L-function.

Ben Salisbury (•) (Central Michigan University) spoke on the *Combinatorics of the Casselman-Shalika formula in type A*. The Casselman-Shalika formula evaluates the spherical Whittaker function in terms of a Weyl character. In recent years, Brubaker-Bump-Friedberg, Bump-Nakasuji, and others have interpreted this formula as a sum over a crystal graph, to which each vertex is attached a polynomial determined by data embedded in the graph. In this talk, he discussed how the tableaux realization of the crystal encodes this same data in a local way; i.e., without the graph structure.

Anne Schilling (University of California, Davis) spoke on joint work with Jennifer Morse on *Weak crystal operators*. Weak crystal operators are crystal operators on the weak order of the affine symmetric group and relate to the combinatorics of cores and k-Schur functions. The k-Schur functions play an important role in Macdonald theory. She explained new combinatorial formulas for Gromov-Witten invariants and fusion coefficients by determining the highest weight elements under the new crystal operators. In the limit as $k \to \infty$ one obtains the usual crystal structure.

Shuichiro Takeda (•) (University of Missouri) spoke on *Metaplectic tensor product*. The notion of the metaplectic tensor product of irreducible admissible representations of the n-fold covering group GL_r $(n > 1)$ over a local field has been developed by Mezo. In this talk, an analogous construction was given for automorphic representations of the covering group over the adeles. This metaplectic tensor product of automorphic representations is compatible with the local metaplectic tensor product of Mezo. Its properties were explained.

Nicolas Templier (•) (Princeton University) spoke on *Singularities and large values of Whittaker functions*. Quantum chaos concerns the behavior of eigenfunctions of the Laplacian on a Riemannian manifold as the eigenvalues approach infinity. In particular, for a hyperbolic surface the conjecture of Iwaniec and Sarnak states that if $\Delta f = \lambda f$ then $\|f\|_2 \ll \lambda^\epsilon \|f\|_\infty$. This talk discussed recent joint work with Brumely bounding a power of the eigenvalue λ by $\|f\|_\infty / \|f\|_2$. The work requires estimation of Whittaker functions and uses Stade's formula in an essential way.

Ian Whitehead (•) (Columbia University) spoke on *Constructing affine Weyl group multiple Dirichlet series*. He described the construction of a multiple Dirichlet series whose group of functional equations is the infinite affine Weyl group $\tilde{A}_3$, and which is uniquely determined by certain axioms. This is part of a project of constructing function field multiple Dirichlet series for all affine Weyl groups. The presentation included background on Weyl group multiple Dirichlet series and the Eisenstein conjecture, as well as a number-theoretic application.

4 Outcome of the Meeting

The workshop was well attended by researchers at all points in their careers, and from countries around the world, including Canada, USA, China, Germany, Israel, Japan, Taiwan, and Switzerland. After the completion of the workshop, the organizers received many positive comments from both senior and junior participants. One aspect participants seemed to appreciate was the pre-workshop survey we referred to above, which led to expository talks on topics selected by the participants.

This workshop was organized on the heels of a thematic semester at ICERM (Providence, RI, USA) in Spring 2013, which was also centered on Whittaker functions. Several participants commented positively on the fact that this workshop together with the ICERM semester and our earlier Banff workshop in Summer 2010 (10w5096) has helped build a community of researchers working on these topics. In particular, several participants were able to continue collaborations and maintain momentum on projects they had begun during the ICERM semester.

Other participants appreciated being exposed to new areas interacting with Whittaker functions, with one participant writing that

> *"the talks were generally of a high quality and gave me insight into parts of the field I had not appreciated enough before."*

Another participant wrote

> *"The workshop exceeded my expectations–and I had high expectations to begin with. I found this particular mathematical community to be exceedingly welcoming, and generous with ideas and results. I was able to make two new connections with researchers that I expect will become collaborators, and this workshop has helped me identify specific research goals that will be of interest to this community."*

Junior participants also benefited from the workshop. One wrote

> *"My thesis project benefited enormously from the week...I think my talk got several experts in the field interested in my work...This will make it much easier for me to initiate future collaborations."*

Another junior participant wrote

> *"I really enjoyed the conference. I especially appreciated having the opportunity to present my joint work. I received useful feedback after my talk that gave me some new ideas for future research directions."*

Based on these comments, we believe that the conference was successful. We also believe that by bringing together people from different fields with different strengths and interests, we have faciliated new collaborations. Because of the success of the workshop, we hope to apply to Banff in the future for another week, perhaps with the intent of running in 2016, so that participants can give updates and progress reports, and so that new junior people can be introduced to this material.

References

[1] Rodney J. Baxter. *Exactly solved models in statistical mechanics*. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], London, 1989. Reprint of the 1982 original.

[2] Philippe Biane, Philippe Bougerol, and Neil O'Connell. Littelmann paths and Brownian paths. *Duke Math. J.*, 130(1):127–167, 2005.

[3] B. Brubaker, D. Bump, and A. Licata. Whittaker functions and Demazure operators. preprint, 2012.

[4] Benjamin Brubaker, Daniel Bump, and Solomon Friedberg. Eisenstein series, crystals, and ice. *Notices Amer. Math. Soc.*, 58(11):1563–1571, 2011.

[5] Alina Bucur and Adrian Diaconu. Moments of quadratic Dirichlet L-functions over rational function fields. *Mosc. Math. J.*, 10(3):485–517, 661, 2010.

[6] Vyjayanthi Chari and Andrew Pressley. *A guide to quantum groups*. Cambridge University Press, Cambridge, 1995. Corrected reprint of the 1994 original.

[7] Ivan Cherednik. Whittaker limits of difference spherical functions. *Int. Math. Res. Not. IMRN*, (20):3793–3842, 2009.

[8] Reda Chhaibi. *Littelmann path model for geometric crystals, Whittaker functions on Lie groups and Brownian motion*. PhD thesis, Université Paris VI, 2013.

[9] A. Gerasimov, D. Lebedev, and S. Oblezin. On a Classical Limit of q-Deformed Whittaker Functions. *Letters in Mathematical Physics*, 100:279–290, June 2012.

[10] Alexander Givental and Yuan-Pin Lee. Quantum K-theory on flag manifolds, finite-difference Toda lattices and quantum groups. *Invent. Math.*, 151(1):193–219, 2003.

[11] M. B. Green, S. D. Miller, J. G. Russo, and P. Vanhove. Eisenstein series for higher-rank groups and string theory amplitudes. *ArXiv e-prints*, April 2010.

[12] Taku Ishii and Takayuki Oda. Calculus of principal series Whittaker functions on $SL(n, \mathbf{R})$. *J. Funct. Anal.*, 266(3):1286–1372, 2014.

[13] Hervé Jacquet. Fonctions de Whittaker associées aux groupes de Chevalley. *Bull. Soc. Math. France*, 95:243–309, 1967.

[14] Masaki Kashiwara. On crystal bases. In *Representations of groups (Banff, AB, 1994)*, volume 16 of *CMS Conf. Proc.*, pages 155–197. Amer. Math. Soc., Providence, RI, 1995.

[15] Bertram Kostant. The solution to a generalized Toda lattice and representation theory. *Adv. in Math.*, 34(3):195–338, 1979.

[16] P. Littelmann. Cones, crystals, and patterns. *Transform. Groups*, 3(2):145–179, 1998.

[17] Freydoon Shahidi. *Eisenstein series and automorphic L-functions*, volume 58 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2010.

[18] Eric Stade. Mellin transforms of $\mathrm{GL}(n, \mathbb{R})$ Whittaker functions. *Amer. J. Math.*, 123(1):121–161, 2001.

[19] E. T. Whittaker and G. N. Watson. *A course of modern analysis*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1996. An introduction to the general theory of infinite processes and of analytic functions; with an account of the principal transcendental functions, Reprint of the fourth (1927) edition.