Focused Research Group Report
Borel complexity and classification of operator systems

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August 10th, 2014 – August 17th, 2014

The Focused Research Group meeting on "Borel complexity and classification of operator systems" constituted the first systematic attempt to study the complexity of the classification problem for operator systems. (An *operator system* is a unital, self-adjoint subspace of the space $B(H)$ of bounded linear operators on a Hilbert space.) We sought to find the complexity of the class of all separable operator systems, as well as identify subclasses of operator systems for which the classification problem is tractable.

It is well known that it is hopeless to attempt to obtain any satisfactory classification, in general, of either single operators or operator algebras on an infinite dimensional Hilbert space. Even for the special case of normal operators, which are completely described using the spectral theorem, it is known that no classification is possible with elementary invariants. Specifically, it is shown in [10] that the invariants arising from the spectral theorem are more complex than arbitrary countable structures. More generally, it has been shown that the classification problem for separable C*-algebras has maximal complexity among all the classification problems that admit the orbits of a Polish group action as a complete invariant.

Because of these results, and because even a finitely generated operator system can encode a significant amount of information about the C*-algebra it generates, it was thought that a satisfactory classification of operator systems would be similarly beyond reach. We confirmed this intuition in the case of separable operator systems, showing that the classification problem in this setting has the same complexity as the classification problem for C*-algebras. (The main result of [9] shows that operator systems are classifiable by the orbits of a Polish group action.)

On the other hand we have shown that the class of *finitely generated* operator systems does in fact admit a satisfactory classification. More precisely, the classification problem for finitely generated operator systems is *smooth*, which means that they can be explicitly classified using only using real numbers as invariants. (For example, in the particular case of the operator system generated by a unitary operator, a very concrete complete invariant is given by the spectrum of the operator, up to a rigid motion of the circle.) This result generalizes a result of Arveson, [2], who classified *finitely represented* operator systems.

The result that isomorphism of finitely generated operator systems is smooth is surprising, since no natural and concrete method is known to tell when two finitely generated operator systems are isomorphic. Instead the result is obtained as a consequence of a more general result that applies to any class of proper metric structures that can be axiomatized in the logic for metric structures. (See [3] for a comprehensive introduction.)

Hoping to obtain positive results beyond finitely generated operator systems, we considered the class of approximately finitely represented (AF) operator systems. This is the class of operator systems that can be obtained as a direct limit of operator systems acting on finite dimensional Hilbert spaces. A natural restriction to impose on such direct systems is that the connecting maps are reduced. This guarantees that the direct system is obtained from a corresponding direct system between the C*-envelopes of the building blocks, which allows us to apply the theory of direct limits of C*-algebras. In particular, the C*-envelope of the an AF operator system is an AF C*-algebra. It is natural to conjecture that AF operator systems can be characterized as the operator systems which have AF C*-envelopes. For these operator systems we considered an Arveson–Bratteli invariant that combines the Bratteli diagram of AF C*-algebras from [4] with the invariant for operator systems acting on a finite dimensional Hilbert space introduced by Arveson in [2]. We believe this should provide a complete invariant for the class of AF operator systems, which would generalize both the classification of AF C*-algebras due to Bratteli–Elliott [4, 8] and the classification of finitely represented operator systems due to Arveson [2]. From the perspective of classification, this would imply that these operator systems can be classified by using countable structures as invariants. This result is the best possible, since a result of Camerlo–Gao from [5] implies that the classification problem for AF C*-algebras has maximal complexity among all the classification problems that admit countable structures as complete invariants.

In the future we plan to study the complexity of the classification problem for *nest algebras* and CSL algebras (CSL stands for "commutative subspace lattice"). Nest algebras are operator algebras introduced by Ringrose as infinite-dimensional generalizations of the algebra of upper triangular matrices [15]. The CSL algebras, which were introduced by Arveson, form a much larger class of operator algebras where the methods of nest algebra theory can still be applied.

A successful classification of nest algebras was obtained by Davidson in [6], building on previous works of Andersen [1] and Larson [12]. However, a similar classification result for CSL algebras is generally believed to be impossible because the perturbation-theoretic arguments at the heart of Davidson's classification theorem do not work for general CSL algebras. We believe that complexity theory can be used to confirm that the classification of CSL algebras is truly harder than the classification of nest algebras.

Finally, a future goal for this collaboration is to obtain a satisfactory classification of "well-behaved" discrete quantum groups, which by duality theory is equivalent to classifying the dual class of compact quantum groups. Compact quantum groups were introduced by Woronowicz in [17, 18] to provide a mathematical foundation for the study of symmetries arising in quantum mechanics. The dual class of discrete quantum groups was subsequently studied in [7, 16]. The theory of discrete and compact quantum groups belong to the theory of locally compact quantum groups, first considered by Kustermans-Vaes [11].

This theory of quantum groups has attracted considerable attention in the recent years. Using a mixture of geometric, algebraic, and functional-analytic techniques, many results for locally compact groups have been generalized to the much broader class of locally compact quantum groups. It would therefore be of great interest to obtain a satisfactory classification for any of several key classes of quantum groups. For example, the full classification of orthogonal easy quantum groups was recently completed in [14], building on previous works of Banica, Bichon, Collins, Curran, Raum, Speicher, and Weber. We plan to consider the class of discrete quantum groups, especially those of Kac type, and show that they admit classification by countable structures. (This is optimal since countable groups already have maximal complexity for countable structures by a result of Mekler [13].) More generally our goal is to study the complexity of the classification problem for quantum groups from the point of view of invariant complexity theory. We believe this will shed new light on the general theory of quantum groups, suggesting which classes of quantum groups are amenable to a satisfactory theory of classification, and which complete invariants may be employed.

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