Spectrum Asymptotics of Operator Pencils

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1 Motivation

Consider the system of equations on a Hilbert space $\mathcal{Z}$

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t). \end{aligned}$$

where A is a closed densely defined operator that generates a strongly continuous semigroup. Also, B and C are bounded operators and u, y are scalar-valued so for some b, $c \in \mathcal{Z}$, $Bu = bu$, $Cx = \langle c, x \rangle$. This is a general framework that includes systems modeled by partial differential or delay-differential equations. In the control system configuration, u is an external control variable and y is an observation. The simplest control law for a system is a constant gain,

$$u(t) = -ky(t) + v(t),$$

where $k > 0$ is real and $v(t)$ is an external signal.

Thus, the spectrum of $A - kBC$ as $k \to \infty$ is of interest in analyzing the dynamics of the controlled system. A plot of these eigenvalues as $k \to \infty$ is known as a root locus plot. More generally, the question of how the spectrum of an operator $A_k = A - kD$ for some fixed operators A, D varies with a parameter k can also be formulated in terms of a root locus. The behavior of the spectrum as $k \to \infty$ is particularly difficult to analyze. Since the behaviour of the root locus, particularly at infinity, can be quite different from that of finite-dimensional approximations, such as finite elements, there are practical applications to theoretical analysis.

2 Overview

Zeros have the same importance in control that eigenvalues do for dynamics.

Definition 1. *The* invariant zeros of $\Sigma(A, B, C)$ *are the set of all* λ *such that*

$$\begin{bmatrix} \lambda I - A & -B \\ C & 0 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{1}$$

has a solution for some scalar u *and non-zero* $x \in D(A)$.

The invariant zeros are the solution to a generalized eigenvalue problem involving an operator pencil.

Extension of the nature of the zeros and of the root locus, well-known for finite-dimensional systems, to infinite-dimensions has been elusive. In [1] the root locus is considered for the case where A is self-adjoint with compact resolvent on a Hilbert space $\mathcal{Z}$, B is a linear bounded operator from $\mathbb{C}^p$ to $\mathcal{Z}$, and $C : D(C) \to \mathbb{C}^p$ where $D(A) \subset D(C)$ is A-bounded. A complete analysis of collocated boundary control of parabolic systems on an interval was provided in [2]. The analysis in that paper uses results from differential equations theory and is difficult to extend to more general classes. In [4] high-gain output feedback of infinite-dimensional systems in the case where A generates an analytic semigroup and $B = C^*$ was studied. The zeros of the system are given as the eigenvalues of an operator and a nonlinear stabilizing feedback law is constructed. Zeros of systems where A is self-adjoint and $B = C^*$ are shown to be real and be bounded by α if $A + A^* \leq 2\alpha I$ on $D(A)$ in [9]. If moreover, the system transfer function can be written in spectral form, and additional technical conditions are satisfied, the poles and the zeros interlace on the real axis.

The problem of defining and analyzing the root locus of infinite-dimensional systems can be formulated as determining the asymptotic spectrum of a class of operator pencils. Unfortunately, these theoretical problems have not been well-studied. We have been working on this problem for a few years and have made some significant progress. We have shown that the root locus for infinite-dimensional systems is well-defined. If no invariant zeros are in the spectrum of A each eigenvalue of A defines a branch of the root locus and these curves are smooth and non-intersecting. Moreover, if any branch converges to a point, that point is a zero of the system. Conversely, each zero is the terminus of a branch of the root locus. If A is self-adjoint and the system is collocated ($b = c$) then the zero interlace with the eigenvalues on the real axis. On an infinite-dimensional space, there are no asymptotes: every branch converges to a zero. (On a finite-dimensional space there is a single asymptote along the negative real axis.) However, large distance from our respective universities and our other duties has made it difficult to obtain a length of uninterrupted time to work on the remaining, difficult, problems. The week at the Banff Centre was amazingly helpful in terms of finding answers to some open problems, and also in discovering some unexpected avenues of future research.

3 Outcome of the Meeting

We have now shown that if A is skew-symmetric and the system is collocated, then all the zeros of the system are imaginary and interlace with the eigenvalues. As the index number increases, the zeros (under certain conditions) become arbitrarily close to the eigenvalues. Also, the entire negative real axis is in the root locus and is an asymptote. Note that the asymptotic behavior of the root locus is very different self-adjoint and skew-adjoint systems.

An unexpected avenue that appeared was the importance of the pseudo-spectrum.

Definition 2. *[8] For any $\epsilon > 0$ the* ϵ-pseudospectrum *of an operator $A : D(A) \subset \mathcal{Z} \to \mathcal{Z}$ is*

$$\sigma_\epsilon(A) = \{s \in \mathbb{C} \mid \|sz - Az\| < \epsilon \text{ for some } z \in D(A), \|z\| = 1\}.$$

In general, the pseudospectrum of an operator can be quite different from its spectrum. However, for normal operators the ϵ-pseudospectrum equals the union of ϵ-balls around the spectrum of A.

Theorem 1. *[8] If A is normal,*

$$\sigma_\epsilon(A) = \bigcup_n B(\lambda_n, \epsilon)$$

where $B(\lambda, \epsilon) := \{s \in \mathbb{C} \mid |s - \lambda| < \epsilon\}$.

Theorem 2. *[6, Thm. 2.3] Suppose that $\Sigma(A, B, C)$ is a system with $\langle b, c\rangle \neq 0$. Define*

$$Kz = \frac{\langle Az, c\rangle}{\langle b, c\rangle}, \quad A_\infty z = Az - bKz, \quad z \in D(A_\infty) = D(K) = D(A). \tag{2}$$

Then, indicating the kernel of C by $c^\perp := \{x \in X \mid \langle x, c\rangle = 0\}$, $(A + bK)(c^\perp \cap D(A)) \subset c^\perp$ and the invariant zeros of $\Sigma(A, B, C)$ are eigenvalues of $A_\infty|_{c^\perp}$. Moreover, denoting by $\{\mu_n\}$ the invariant zeros of $\Sigma(A, B, C)$, the corresponding eigenfunctions of $A_\infty|_{c^\perp}$ are given by $\{(\mu_n I - A)^{-1} b\}$.

Theorem 2 generalizes to systems for which $\langle b, c\rangle = 0$ [6, 7].

The following theorem shows that under certain assumptions, the zeros are asymptotically close to the pseudospectrum of A. A sequence $\{\phi_n\}$ in $\mathcal{Z}$ is called a *Riesz system* in $\mathcal{Z}$ if there exists an isomorphism $S \in L(\mathcal{Z})$ such that $\{S\phi_n\}$ is an orthonormal system in $\mathcal{Z}$.

Theorem 3. *Suppose that $\Sigma(A, B, C)$ is a system with $\langle b, c\rangle \neq 0$, the eigenfunctions of A_∞, see (2), corresponding to the invariant zeros of $\Sigma(A, B, C)$ form a Riesz system, and $c \in D(A^*)$. Write the invariant zeros of $\Sigma(A, B, C)$ as $\{\mu_1, \mu_2, \ldots\}$ (repeated according to multiplicity) and indicate the corresponding eigenfunctions of A_∞ by $\{z_1, z_2, \ldots\}$. Recall that the eigenvalues of A are indicated by $\{\lambda_n\}$. Then for any $\epsilon > 0$ there is N so that for all $n > N$*

$$\|Az_n - \mu_n z_n\| < \epsilon,$$

that is, $\mu_n \in \sigma_\epsilon(A)$.

This is used to show that in many cases the zeros become asymptotically close to the eigenvalues. The results described briefly here have been submitted to a major journal [3] and we plan we present our work at a large conference next year.

4 Future Research Directions

The application of the pseudo-spectrum to the analysis of zeros is new. It now appears that the pseudo-spectrum can be very useful in analyzing the stability of control systems. Unfortunately, little is known about the pseudo-spectrum for operators. Do operators with a Riesz basis have pseudo-spectrum equal to ϵ-spectrum balls? Is the root locus is in the ϵ-spectrum balls around the eigenvalues?

Also, consider a more general spectral problem than (1):

$$\beta \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} z = \alpha \begin{bmatrix} A & -B \\ C & 0 \end{bmatrix} z$$

For $\beta \neq 0$, $\lambda = \frac{\alpha}{\beta}$ is the invariant zeros. But $\beta = 0$ (or $\lambda = \infty$) is also an eigenvalue. The root locus of delay, self-adjoint and skew-adjoint cases have totally different behavior at infinity. Does pseudo-spectrum of this pencil provide any insight into the asymptotic behavior of the spectrum of operator pencils?

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