# Dirichlet spaces and de Branges–Rovnyak spaces

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## 1 Overview of the Field

The spaces mentioned in the title are two interesting families of Hilbert spaces of holomorphic functions, both contained inside $H^2$, the classical Hardy space on the unit disk $\mathbb{D}$.

### 1.1 Weighted Dirichlet spaces

Given a non-negative function $\omega \in L^1(\mathbb{D}, dA)$ and $f \in H^2$, we define the weighted Dirichlet integral

$$\mathcal{D}_\omega(f) := \frac{1}{\pi}\int_{\mathbb{D}} |f'(z)|^2\,\omega(z)\,dA(z).$$

The weighted Dirichlet space $\mathcal{D}_\omega$ is the set of $f \in H^2$ such that $\mathcal{D}_\omega(f) < \infty$. It is a Hilbert space with respect to the norm $\|\cdot\|_{\mathcal{D}_\omega}$ defined by $\|f\|^2_{\mathcal{D}_\omega} := \|f\|^2_{H^2} + \mathcal{D}_\omega(f)$.

The case of primary interest is where $\omega$ is a positive superharmonic function on $\mathbb{D}$. In this case, there is a unique positive finite measure $\mu$ on $\overline{\mathbb{D}}$ such that

$$\omega(z) = \int_{\mathbb{D}} \log\Big|\frac{1-\overline{\zeta}z}{\zeta - z}\Big|\frac{1}{1-|\zeta|^2}\,d\mu(\zeta) + \int_{\mathbb{T}} \frac{1-|z|^2}{|\zeta - z|^2}\,d\mu(\zeta) \qquad (z \in \mathbb{D}).$$

We then write $\mathcal{D}_\mu$ for $\mathcal{D}_\omega$. It can be shown that, if $f \in \mathcal{D}_\mu$, then $f$ has radial limits $\mu$-a.e. on $\mathbb{T}$, and

$$\mathcal{D}_\mu(f) = \int_{\overline{\mathbb{D}}}\int_{\mathbb{T}}\Big|\frac{f(\lambda)-f(\zeta)}{\lambda-\zeta}\Big|^2\,\frac{|d\lambda|}{2\pi}\,d\mu(\zeta). \tag{1}$$

The classical Dirichlet space is obtained by taking $\mu$ to be normalized Lebesgue measure on $\mathbb{T}$. When $\mu$ is a general measure on $\mathbb{T}$, we obtain the harmonically weighted Dirichlet spaces, first introduced by Richter [9] as part of his analysis of closed shift-invariant subspaces of the classical Dirichlet space, and subsequently studied by Richter and Sundberg in [10] (see also [6, Chapter 7]). The study of general superharmonic weights was initiated by Aleman [1]. They have the advantage of including both the harmonic weights and the important family of radial weights $\omega(z) = (1-|z|^2)^\alpha$ for $0 < \alpha < 1$.

### 1.2 De Branges–Rovnyak spaces

Let $b$ be a holomorphic function on $\mathbb{D}$ such that $|b| \leq 1$. The de Branges–Rovnyak space $\mathcal{H}_b$ is the Hilbert space of holomorphic functions on $\mathbb{D}$ with reproducing kernel $(1 - \overline{b}(w)b(z))/(1 - \overline{w}z)$.

If $b$ is an inner function, then $\mathcal{H}_b = H^2 \ominus bH^2$, a closed subspace of $H^2$ often called a model subspace.

In this project, we are interested exclusively in the 'opposite' case, namely when $\log(1 - |b|^2) \in L^1(\mathbb{T})$. This condition is equivalent to $b$ being a non-extreme point of the unit ball of $H^\infty$. It is also equivalent to $\mathcal{H}_b$ containing the polynomials. Henceforth, we always assume that $b$ satisfies this condition.

Under this hypothesis on $b$, there is a unique outer function $a$ such that $a(0) > 0$ and $|a|^2 + |b|^2 = 1$ a.e. on $\mathbb{T}$. It can be shown that $f \in \mathcal{H}_b$ iff $T_{\overline{b}} f \in T_{\overline{a}}(H^2)$, where $T_{\overline{a}}, T_{\overline{b}}$ denote the Toeplitz operators associated to $\overline{a}, \overline{b}$ respectively. In this case there is a unique function $f^+ \in H^2$ such that $T_{\overline{b}} f = T_{\overline{a}}(f^+)$, and

$$\|f\|^2_{\mathcal{H}_b} = \|f\|^2_{H^2} + \|f^+\|^2_{H^2}. \tag{2}$$

We write $\phi := b/a$. Note that $\phi \in \mathcal{N}^+$, the Smirnov class. Conversely, every $\phi \in \mathcal{N}^+$ is of the form $\phi = b/a$ for a unique pair $(b, a)$ as above. Thus, specifying $\phi$ is equivalent to specifying $b$, and it is often more convenient to work in terms of $\phi$.

The spaces $\mathcal{H}_b$ were introduced by de Branges and Rovnyak in the appendix of [4] and further studied in [5]. The initial motivation was to provide canonical model spaces for certain types of contractions on Hilbert spaces. Subsequently it was realized that these spaces have numerous connections with other topics in complex analysis and operator theory, in particular through Toeplitz operators. For further information on this topic, we refer to the books of de Branges and Rovnyak [5], Sarason [11], and the forthcoming monograph of Fricain and Mashreghi [7].

## 2 Recent Developments and Open Problems

Sarason [12] discovered a strong connection between certain Dirichlet spaces and de Branges–Rovnyak spaces. He showed that, if $\mu$ is the Dirac mass at a point $\zeta \in \mathbb{T}$, then the Dirichlet space $\mathcal{D}_\mu$ is isometrically equal to the de Branges–Rovnyak space $\mathcal{H}_b$, where $b$ corresponds to the function $\phi(z) := z/(1 - \overline{\zeta}z)$. For this choice of $\phi$, it is quite easy to see that dilation is a contraction on $\mathcal{H}_b$, in other words $\|f_r\|_{\mathcal{H}_b} \leq \|f\|_{\mathcal{H}_b}$, where $f_r(z) := f(rz)$. Sarason used this to deduce that dilation is also a contraction on $\mathcal{D}_\mu$.

The authors of [2] obtained a converse to Sarason's result: the only measures $\mu$ on $\mathbb{T}$ for which $\mathcal{D}_\mu$ is isometrically equal to some $\mathcal{H}_b$ are point masses. The proof made use of a formula expressing the norm of certain functions in $\mathcal{H}_b$ in terms of their Taylor coefficients and those of $\phi$: if $f$ is holomorphic in a neighbourhood of $\overline{\mathbb{D}}$, then $\sum_{j\geq 0} \widehat{f}(j+k)\overline{\widehat{\phi}(j)}$ converges absolutely for each $k$, and

$$\|f\|^2_{\mathcal{H}_b} = \sum_{k\geq 0} |\widehat{f}(k)|^2 + \sum_{k\geq 0} \Big|\sum_{j\geq 0} \widehat{f}(j+k)\overline{\widehat{\phi}(j)}\Big|^2. \tag{3}$$

It was left open whether this same formula is valid for all $f \in \mathcal{H}_b$.

It was also shown in [2] that, for certain $b$, dilation is no longer a contraction on $\mathcal{H}_b$, and even that $\limsup_{r\to 1} \|f_r\|_{\mathcal{H}_b} = \infty$ for some $f \in \mathcal{H}_b$. It was left open whether $\limsup$ can be replaced by $\liminf$. This was of interest because the only proofs that polynomials are dense in $\mathcal{H}_b$ were non-constructive, and knowing that we always have $\liminf_{r\to 1} \|f_r\|_{\mathcal{H}_b} < \infty$ would open the door to the construction of polynomial approximants to $f$.

The question of when $\mathcal{D}_\mu$ is isomorphically equal to $\mathcal{H}_b$ (as opposed to isometrically equal) was raised in [2] and studied in [3] for the case of measures $\mu$ on $\mathbb{T}$. It was shown that, in order for $\mathcal{D}_\mu$ to be isomorphically equal to some $\mathcal{H}_b$, it is necessary that $\mu$ be singular with respect to Lebesgue measure on $\mathbb{T}$, and sufficient that $\mu$ have finite support. It was left open whether there are any examples of $\mu$ with infinite support.

In the classical Dirichlet space, it is known that every function has tangential limits at almost every point of the unit circle, where the tangential approach region at $\zeta$ is of the form $|z - \zeta| = O(|\log(1 - |z|)|)$ (see [8]). It is still an open problem to determine the optimal approach region for $\mathcal{D}_\mu$ for general $\mu$.

# 3 Scientific Progress Made

## 3.1 Dilation in $\mathcal{H}_b$

We solved affirmatively the problem of whether $\limsup$ can be replaced by $\liminf$.

**Theorem 3.1.** *There exist $b$ and $f \in \mathcal{H}_b$ such that $\|f_r\|_{\mathcal{H}_b} \to \infty$ as $r \to 1$.*

The proof actually gives a little more, namely an example such that $|(f_r)^+(0)| \to \infty$ as $r \to 1$. From this, it is a simple matter to deduce that formula (3) does **not** hold for general $f \in \mathcal{H}_b$. Indeed:

**Corollary 3.2.** *There exist $b$ and $f \in \mathcal{H}_b$ such that $\sum_{j\geq 0} \widehat{f}(j)\overline{\widehat{\phi}(j)}$ diverges.*

## 3.2 Polynomial approximation in $\mathcal{H}_b$

Though the preceding theorem kills off the possibility of obtaining polynomial approximants in $\mathcal{H}_b$ via dilations, we did find another method for constructing such approximants, based on the following result.

**Theorem 3.3.** *Let $(\psi_n)$ be a sequence in $H^\infty$ such that $\|\psi_n\|_{H^\infty} \to 1$ and $\psi_n(0) \to 1$. Then, for all $f \in \mathcal{H}_b$, we have $\|T_{\overline{\psi}_n} f - f\|_{\mathcal{H}_b} \to 0$.*

If $p$ is a polynomial, then so is $T_{\overline{\psi}_n} p$. If, in addition, $\psi_n \in aH^\infty$, then $T_{\overline{\psi}_n}$ is a bounded operator from $H^2$ into $\mathcal{H}_b$ with norm at most $\|\psi_n/a\|_{H^\infty}$. Combined with the theorem, this leads to a constructive proof of

**Corollary 3.4.** *Polynomials are dense in $\mathcal{H}_b$.*

## 3.3 Isometric equality between $\mathcal{D}_\mu$ and $\mathcal{H}_b$

We extended the results of [12] and [2] from harmonic weights to superharmonic weights.

**Theorem 3.5.** *Let $\mu$ be a finite positive measure on $\overline{\mathbb{D}}$ and let $\phi \in \mathcal{N}^+$ (with corresponding $b$). Then $\mathcal{D}_\mu = \mathcal{H}_b$ with equality of norms if and only if there exist $\zeta \in \overline{\mathbb{D}}$ and $\alpha \in \mathbb{C}$ such that*

$$\mu = |\alpha|^2 \delta_\zeta \qquad \text{and} \qquad \phi(z) = \alpha z/(1 - \overline{\zeta} z).$$

The 'if' part of the theorem leads quickly to the following corollary.

**Corollary 3.6.** *If $\mu$ is a finite positive measure on $\overline{\mathbb{D}}$, then*

$$\mathcal{D}_\mu(f_r) \leq \frac{2r}{1+r} \mathcal{D}_\mu(f) \qquad (0 < r < 1).$$

## 3.4 Isomorphic equality between $\mathcal{D}_\mu$ and $\mathcal{H}_b$

With the ultimate aim of constructing new examples of pairs $(\mu, b)$ with $\mathcal{D}_\mu = \mathcal{H}_b$ isomorphically, we studied a family of examples of $\mathcal{H}_b$-spaces for which it is possible to compute the norm exactly.

**Theorem 3.7.** *Let $\nu$ be a complex measure on $\overline{\mathbb{D}}$, and let $\phi(z) := z \int (1 - \overline{\zeta} z)^{-1} \, d\nu(\zeta)$ $(z \in \mathbb{D})$. Then, for all $f$ holomorphic in a neighborhood of $\overline{\mathbb{D}}$, we have*

$$\|f\|^2_{\mathcal{H}_b} = \|f\|^2_{H^2} + \int_{\mathbb{T}} \Big| \int_{\overline{\mathbb{D}}} \frac{f(\lambda) - f(\zeta)}{\lambda - \zeta} \, d\nu(\zeta) \Big|^2 \frac{|d\lambda|}{2\pi}.$$

*If, in addition, $\nu \ll \mu$ and $d\nu/d\mu \in L^2(\mu)$, then $\mathcal{D}_\mu \subset \mathcal{H}_b$.*

## 3.5 Tangential approach regions for $\mathcal{D}_\mu$

We had just enough time for a summary discussion of this problem. It seems likely that the optimal approach region should be expressed in terms of the reproducing kernel of $\mathcal{D}_\mu$. Although there is no explicit expression for this kernel, there are now precise estimates for its norm. We intend to return to this problem.

# 4 Outcome of the Meeting

The results obtained have now been written up as a detailed report with a view to eventual publication. It is a pleasure to thank BIRS for the opportunity to advance our work in such a pleasant setting. We also acknowledge with thanks financial support from the UMI-CRM.

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