Operator limits of random matrices

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The main objective of the workshop was to complete a joint project of the two participants on operator limits of certain random matrices.

1 Background and Motivation

Wigner introduced the Gaussian ensembles in the 1950s to model large atomic nuclei, whose energy levels were eigenvalues of such a complex system that it may as well be taken to be random. They exhibited repulsion, unlike independent random points. Dyson found that their joint density is given by

$$\frac{1}{Z_{n,\beta}} e^{-\frac{\beta}{4}\sum_{k=1}^{n}\lambda_k^2} \prod_{1\le j<k\le n} |\lambda_j-\lambda_k|^\beta, \tag{1}$$

where $\beta = 1, 2, 4$ correspond to real symmetric and complex Hermitian, and self-dual quaternion Gaussian matrices. In order to distill an essential random eigenvalue process, one takes a limit as $n \to \infty$. For $\beta = 1, 2, 4$, the Gaudin-Mehta theorem (see e.g. [6]) shows that in the bulk there is a point process limit. While the finite n distribution dates back to 1962, the general beta point process limit was first shown in the participants' paper [11]; these are the Sine_β processes.

These point processes (especially in the $\beta = 2$ case) show up as limits in various places in mathematics. One of the most famous examples is the Montgomery-Dyson conjecture [7], which states that y-coordinates of the critical zeros of the Riemann ζ-function, i.e. the set $\mathcal{Z} = \{y : \zeta(1/2 + iy) = 0\}$, 'looks like' the Sine_2 process. To be more precise: if U is uniform on $[0, 1]$ then $(\mathcal{Z} - t \cdot U) \log t$ is supposed to converge in distribution to the point process Sine_2.

The Hilbert-Pólya conjecture is an approach to proving the Riemann hypothesis – the goal is to find a self-adjoint operator whose zero set is the same as that of $\mathcal{Z}$. The natural random matrix version of the question (attributed to Sarnak) is whether there is a natural random self-adjoint operator whose zero set is the Sine_2 process.

The study of the general Gaussian β-ensembles got a big boost from the work of Dumitriu and Edelman [1] who introduced a random tridiagonal matrix model with joint eigenvalue density given by (1). Their paper provided a tridiagonal representation for another related beta-ensemble, and similar representations appeared in [3] for beta generalizations of other classical random matrix ensembles. Edelman and Sutton [10], [2] conjectured that in the appropriate scaling limit these tridiagonal matrix models converge to certain random differential operators, and the point process limits of the beta ensembles are given by the spectra of these operators.

These predictions were confirmed rigorously for the so-called soft edge and hard edge limit scaling limits in [9] and [8] . In both cases one can show that the appropriate scaling limits of the tridiagonal random matrix models are given by certain second order differential operators with random potential and these operators have an a.s. discrete spectrum (which give the limiting point processes).

The starting point for our research project was to find a similar representation for the Sine_β processes (the bulk scaling limits), for the $\beta = 2$ case this would resolve the question posed by Sarnak.

2 Recent Developments and Open Problems

Recently we managed to find an random operator representation for the Sine_β process. The operator in question is a first order two-dimensional differential operator, it fits into the general framework of the classical Dirac operators. In order to define it, one first needs the hyperbolic Brownian motion $\mathcal{B}$ satisfying the SDE

$$d\mathcal{B} = (1 - |\mathcal{B}|^2)dZ, \qquad \mathcal{B}_0 = 0,$$

where $Z = Z_1 + iZ_2$ with independent real standard Brownian motions Z_1, Z_2. Let

$$X_t = \frac{1}{\sqrt{1-|\mathcal{B}_t|^2}} \begin{pmatrix} 1 & \mathcal{B}_t \\ \overline{\mathcal{B}_t} & 1 \end{pmatrix}, \qquad J = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \qquad \tau(t) = -\beta^{-1} \log(1-t).$$

Then the differential operator

$$2JX^2_{\tau(t)}\,\partial_t,$$

acting on functions $[0, 1] \to \mathbb{C}^2$ with appropriate boundary and L_2 conditions has an a.s. discrete spectrum which is distributed as the Sine_β process. (Note that the β dependence only appears via the time change function τ.)

The goal of the workshop was to explore various other random matrix models to see whether similar Dirac operator representations appear in other places. We also wanted to study how the found random operator could be used to prove limit theorems for the finite beta ensembles. Finally, we hoped to unify the descriptions of the soft edge, hard edge and bulk limits of β-ensembles.

3 Scientific Progress Made

We found the workshop to be extremely fruitful, and we have made significant progress in our project. Here is a partial list of the results of the meeting.

- We understood the connection between the so-called carousel description of the Sine_β process given in [11] and the oscillation theory of Dirac operators. This also allowed us to give an alternative description of the hard edge limit operator given in [8] as a random Dirac operator, unifying the descriptions of the hard edge and bulk cases.
- We set up the framework for a robust method for proving operator level scaling limits for random tridiagonal and random CMV matrices. It relies on the observation that the inverse of Dirac operators are Hilbert-Schmidt integral operators, and for many of the finite models one can recover a discrete approximation of the limiting integral operator in the finite systems. We apply the method to study the operator level convergence for the Gaussian β-ensemble given in (1), the circular β-ensemble (for which the point process limit was proved in [4]) and for a family of random discrete Schrödinger operators studied in [5]. As a consequence we gave a proof for the fact that the limit of the circular β-ensembles and the bulk limit of the Gaussian β-ensembles are the same.
- We described how the soft edge limiting operator appears as a limit of the hard edge operators (this is the so-called hard-to-soft transition).

4 Outcome of the Meeting

We are currently in the process of writing up our results. One of the participants (Virág) is an invited speaker at the 2014 International Congress of Mathematics. Some of the results of the workshop will also appear in his contribution to the Proceedings of ICM [12].

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