Even Symmetry of Axially Symmetric Solutions of Allen-Cahn Equation

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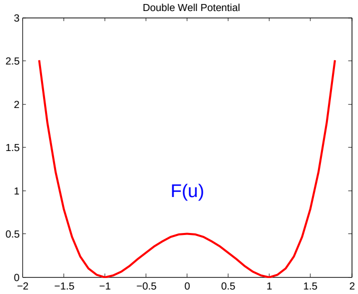
Allen-Cahn Equation in Entire Space

$$\Delta u - F'(u) = 0, \quad |u| < 1, \quad x \in \mathbb{R}^n. \tag{1}$$

where F is a double well potential.

Example:

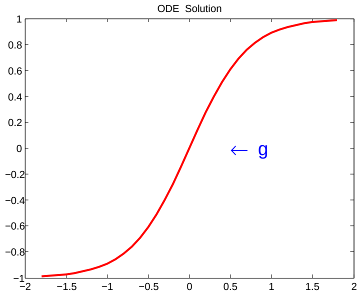
$$F(u) = \frac{1}{4}(1 - u^2)^2, \quad u \in \mathbb{R}.$$



ODE Solution and Transition Profile

The ODE solution g to Allen-Cahn equation is unique up to translation.

$$\begin{cases} g''(s) - F'(g(s)) = 0, & s \in \mathbb{R}, \\ \lim_{s\to\infty} g(s) = 1, & \lim_{s\to-\infty} g(s) = -1. \end{cases} \tag{2}$$



Minimizing and Nondegeneracy of g

We may assume that $g(0) = 0$. Indeed, g is a minimizer of the following energy functional

$$E(v) := \int_{-\infty}^{\infty} [\frac{1}{2}|v'|^2 + F(v)]dx$$

in $\mathcal{H} := \{v \in H^1_{loc}(\mathbb{R}) : -1 \leq v \leq 1,\ \lim_{s\pm\infty} v(s) = \pm 1\}$ and

$$\mathbf{e} := E(g) = \int_{-1}^{1} \sqrt{2F(u)}du < \infty.$$

The solution g is non-degenerate in the sense that the linearized operator has a kernel spanned only by g'.

De Giorgi Conjecture

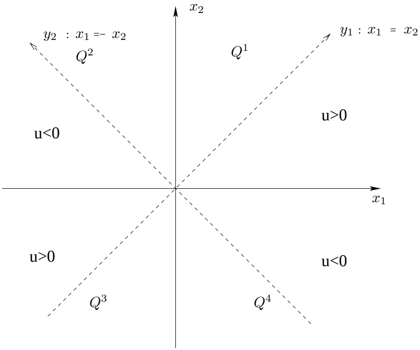
Conjecture (De Giorgi, 78)

If u satisfies (1) and a monotone condition

$$u_{x_n}(x) > 0, \quad x \in \mathbb{R}^n, \tag{3}$$

then for at least $n \leq 8$, u must be a 1-d solution, i.e. a proper extension, rotation and translation of g. In other words, the level sets of u must be hyper planes.

Example of Saddle Solution



Existence of saddle solution

Theorem

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iv) *(Gui, Liu and Wei, 2014) Variational characterization of saddle solutions with four ends (as in Liu's talk).*

A Problem in a half plane

Let $\mathbb{R}^2_+ := \{(x,y) | x > 0,\ y \in \mathbb{R}\}$. Consider

$$\begin{cases} u_{xx} + u_{yy} - F'(u) = 0, \quad |u| < 1, \quad (x,y) \in \mathbb{R}^2_+ \\ u_x(0,y) = 0, \quad u_x(x,y) > 0, \quad (x,y) \in \mathbb{R}^2_+. \end{cases} \tag{5}$$

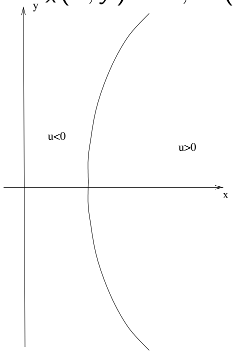

Figure : Saddle Solution

Even symmetry in y and asymptotical behavior

Theorem (Gui, 09)

Assume that $u(x, y) = u(-x, y)$ *satisfies* (5).

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$$u(x, y) = u(x, -y), \quad (x, y) \in \mathbb{R}^2 \tag{6}$$

after a proper translation in y.

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after a proper translation in y.

Moreover, the 0-level set of u *for* $x > x_0$ *can be expressed as the graph of two* $C^{3,\beta}$ *functions* $y = \pm k(x)$, *where* $k(x)$ *satisfies*

$$k(x) = \kappa x + C + o(1) \tag{7}$$

some constants $\kappa > 0, C$.

Energy estimate

Lemma

Suppose that u is a solution to (5). Then

$$\int_{\mathbb{R}} [F(u(0,y)) + \frac{1}{2} u_y^2(0,y)] dy < 3\mathbf{e}. \qquad (8)$$

Define

$$h(y) = \int_0^\infty u_y u_x dx, \quad \forall y \in \mathbb{R}.$$

In view of the positivity of u_x, it is easy to see that $h(y)$ is well-defined and

$$|h(y)| < \int_0^\infty \sqrt{2F(u(x,y))} \cdot u_x dx < \mathbf{e}, \quad \forall y \in \mathbb{R}.$$

Differentiating $h(y)$ with respect to y and using (5), we can prove the lemma.

Hamiltonian identities

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$$E(x) = \int_{\mathbb{R}} y[F(u(x,y)) + \frac{1}{2}u_y^2(x,y) - \frac{1}{2}u_x^2(x,y)]dy \equiv C. \qquad (10)$$

Preliminary Analysis of Level Set

The 0-level set Γ of u can be regarded as the graph of a function $x = \gamma(y)$ which is defined for $y \leq K_1$, and $y \geq K_2$ with $K_1 \leq K_2$ and is C^3.

Lemma

There exists $\kappa > 0$ such that

$$\lim_{y \to \infty} \gamma'(y) = \kappa, \quad \lim_{y \to -\infty} \gamma'(y) = -\kappa. \tag{11}$$

It is important to exclude $\kappa = 0$ and $\kappa = \infty$.

Level Set are Asymptotically Straight Lines

Lemma (Gui, 06)

Suppose u is a solution to (1) in $\mathbb{R}^2$ with finite Morse index, and the level set of u in a cone $\mathcal{C}_\alpha := \{(x, y) : |y| < (\tan \alpha)x\}$ for $x > K > 0$ is a smooth curve $y = \mathcal{L}(x)$ which indeed is contained in a smaller cone $\mathcal{C}_\beta := \{(x, y) : |y| < (\tan \beta)x\}$ for $x > K > 0$ with $0 < \beta < \alpha < \pi/2$.
Then

$$\mathcal{L}(x) = kx + A + o(1), \quad as \ x \to \infty$$

for some constant $k \in (-\tan \beta, \tan \beta)$.

Entire solutions with $2k$ ends

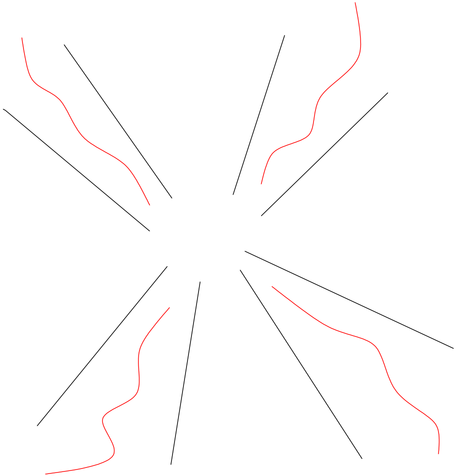

Figure : Level Set with Four Ends

Symmetry of Saddle Solution with Four Ends

Theorem (Gui, 09)

Suppose that u is a saddle solution to (1) in $\mathbb{R}^2$ with four ends (two 0-level curves), then after proper translation and rotation u satisfies:

$$u(x,y) = u(x,-y) = u(-x,y), \quad \forall (x,y) \in \mathbb{R}^2 \tag{12}$$

and

$$u_x > 0, \quad u_y < 0, \quad \forall x > 0,\ y > 0 \tag{13}$$

Remark

Kowalzyck, Liu and Pacard showed the existence of four end solutions for all contact angle $\theta \in (0, \pi/2)$, using the symmetry result and monotone results shown above.

Ideas of Proof

- Asymptotically behavior of u;
- The Moving Plane Method;
- Hamiltonian identities;

Axially Symmetric Solutions in $\mathbb{R}^n$, $n \geq 3$

If u is an axially symmetric solution, i.e., $u(x, y) = u(|x|, y)$, then (1) can be written as

$$u_{rr} + \frac{n-2}{r} u_r + u_{yy} - F'(u) = 0, \ |u| < 1, \quad r = |x|, \ (x, y) \in \mathbb{R}^n. \tag{14}$$

We note that $u_r(0, y) = 0, \forall y \in \mathbb{R}$.
We may assume the monotone condition

$$u_r(r, y) > 0, \quad r = |x| > 0, \ y \in \mathbb{R}. \tag{15}$$

Existence of Axially Symmetric Solution in $\mathbb{R}^n, n = 3$

Using Catenoids in $\mathbb{R}^n, n = 3$, Del Pino, Kowalczyk and Wei showed the existence of such solutions.

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It is expected that similar solutions exist for $\mathbb{R}^n, n \geq 4$ (communications with Del Pino, Wei).

Main Theorem

Assume that $u(x, y) = u(|x|, y)$ satisfies (14) and (15). Then

$$u(r, y) = u(r, -y), \quad r = |x| > 0, \ y \in \mathbb{R} \tag{16}$$

after a proper translation in y.

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after a proper translation in y. Moreover, the 0-level set of u for $r > r_0$ can be expressed as the graph of two $C^{3,\beta}$ functions $y = \pm k(r)$, where $k(r)$ satisfies

$$k(r) = \begin{cases} \dfrac{\kappa+2}{2\mu} \ln r + C + o(1), & \text{when } n = 3; \\ \dfrac{1}{\mu} \ln r + \dfrac{1}{2\mu} \ln(\dfrac{\mu A}{n-3}) + o(1), & \text{when } n > 3; \end{cases}$$

for some constants $\kappa > 0$, C, $\mu = \sqrt{F''(1)}$, A.

Lemma

Suppose that u is a solution to (14), (15). *Then*

$$\int_{\mathbb{R}} [\int_0^\infty \frac{n-2}{r} u_r^2 dr] dy + \int_{\mathbb{R}} [F(u(0,y)) + \frac{1}{2} u_y^2(0,y)] dy < 3\mathbf{e}. \tag{17}$$

Preliminary analysis of level set for $n \geq 3$

Let $r = \gamma(y)$ be the nodal curve for $|y|$ large, then

$$\lim_{y \to -\infty} \gamma'(y) = -\infty, \quad \lim_{y \to \infty} \gamma'(y) = \infty. \tag{18}$$

It is more convenient to write the level set as the graph of functions $y = k_1(r)$, $y = k_2(r)$ for $r > R_0$. We have

$$\begin{cases} k_1'(r) < 0, \quad k_2'(r) > 0, \quad \text{for } \; r > R_0 \\ \lim_{r\to\infty} k_1(r) = -\infty, \quad \lim_{r\to\infty} k_2(r) = \infty \\ \lim_{y\to\infty} k_1'(r) = \lim_{r\to\infty} k_2'(r) = 0 \end{cases} \tag{19}$$

Basic Profile

$$
\begin{cases}
\phi'' - F'(\phi) = 0, \quad y > l_2, \text{ or } y < l_1, \text{ or } y \in (l_1, l_2) \\
\phi(l_1) = \phi(l_2) = 0, \quad \lim_{y \to \pm\infty} \phi(y) = -1 \\
\phi(y) > 0, \quad y \in (l_1, l_2)
\end{cases} \tag{20}
$$

Optimal Match

For $r > R_1$ sufficiently large, we can choose a unique pair $l_1(r) < l_2(r)$ so that

$$||u(\cdot, r) - \phi(l_1(r), l_2(r), \cdot)||_{L^2(\mathbb{R})} = \inf_{l_1 < l_2} \{||u(\cdot, r) - \phi(l_1, l_2, \cdot)||\}.$$

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Let $l(r) = (l_2(r) - l_1(r))/2$, then

$$\begin{cases} \frac{n-2}{r} l_1'(1 + o(1)) + l_1'' = -Ae^{-2\mu l(r)}(1 + o(1)) \\ \frac{n-2}{r} l_2'(1 + o(1)) + l_2'' = Ae^{-2\mu l(r)}(1 + o(1)). \end{cases}$$

an ODE

$I(r)$ can be transformed to $W(s)$ which satisfies

$$W''+(n-3)W'+2(n-3)W = \frac{(W')^2}{W}+2\mu A(1+o(1)), \quad W>0, \ \forall s$$

Note $Q(r) = e^{\mu I(r)}$ and $W(s) = r^{-2}Q(r), \quad r = e^s, \quad s \geq s_0$.
In general, consider

$$W'' + \beta_1 W' + \beta_2 W = \frac{(W')^2}{W} + (1 + o(1)), \quad W > 0, \ \forall s > s_0.$$

for some constants β_1, β_2.

A technical lemma

Lemma

Suppose that W is a solution to (26).
(a) If $\beta_1 = \beta_2 = 0$ then there exists positive constants $C, \kappa > 0$ such that

$$W(s) = Ce^{\kappa s}(1 + o(1)), \quad as \quad s \to \infty. \tag{21}$$

(b) If $\beta_1 > 0, \beta_2 > 0$, then

$$W(s) = \frac{1}{\beta_2} + o(1), \quad as \quad s \to \infty. \tag{22}$$

Asymptotic formulas

We derive

$$l_2(r) = \begin{cases} \dfrac{\kappa+2}{2\mu}\ln r + C_2 + o(1), & \text{when } n = 3; \\ \dfrac{1}{\mu}\ln r + \dfrac{1}{2\mu}\ln(\dfrac{\mu A}{n-3}) + C_2 + o(1), & \text{when } n > 3; \end{cases}$$

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Similar formula for $l_1(r)$ with opposite sign for the logarithmic terms.

Ideas of Proof of the Main Theorem

- Asymptotically behavior of u;
- The Moving Plane Method;

Thank You