Liouville theorems and qualitative properties of solutions to competitive systems with several components

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Classification of entire solutions

We are concerned with the classification of positive or nonnegative entire solutions with algebraic growth of the competition-diffusion elliptic system

$$\Delta u_i = \sum_{\substack{j=1 \\ j\neq i}}^{k} u_j^2 u_i \qquad \text{in } \mathbb{R}^N, \text{ for } i = 1, \ldots, k, \qquad \text{(Sys(k))}$$

with $k \geq 2$.

- By positive solution we mean that $u_i > 0$ in $\mathbb{R}^N$ for every i, while by nonnegative solution we admit the possibility that some u_i vanish identically, requiring however that at least two components are non-trivial.
- By the strong maximum principle, if $u_i \geq 0$ and $u_i \not\equiv 0$, then $u_i > 0$ in $\mathbb{R}^N$.
- A trivial solution is when all components vanish but possibly one, which is constant.

The following Theorem was the key result to prove uniform a priori bounds in Hölder spaces for solutions to reaction-diffusion systems with large competition, satisfying an initial L^∞-bound.

Theorem (B. Noris, H. Tavares, S.T., G. Verzini 2010)

Let $k \geq 2$, and let $\mathbf{u} = (u_1, \ldots, u_k)$ be a nonnegative solution of

$$\Delta u_i = \sum_{j=1,\, j\neq i}^{k} u_j^2 u_i \qquad \text{in } \mathbb{R}^N, \text{ for } i = 1, \ldots, k.$$

Assume that, for some $\alpha \in (0,1)$, there holds

$$u_1(x) + \cdots + u_k(x) \leq C(1 + |x|^\alpha) \qquad \text{for every } x \in \mathbb{R}^N.$$

Then all components (but possibly one) vanish.

The 2 components system

When $k = 2$

$$\begin{cases} \Delta u = uv^2 & \text{in } \mathbb{R}^N \\ \Delta v = u^2 v & \text{in } \mathbb{R}^N \\ u, v > 0 & \text{in } \mathbb{R}^N \end{cases} \qquad (\mathrm{Sys}(2))$$

This case has been widely investigated in recent years. It appears in the analysis of phase separation phenomena in a binary mixture of Bose-Einstein condensates with multiple states; we refer to the papers by H. Berestycki, T.-C. Lin, J. Wei and C. Zhao, and by H. Berestycki, S.T. K. Wang, J. Wei.

One dimensional solutions

Of course, there are one variable solutions (depending on the energy parameter $h > 0$):

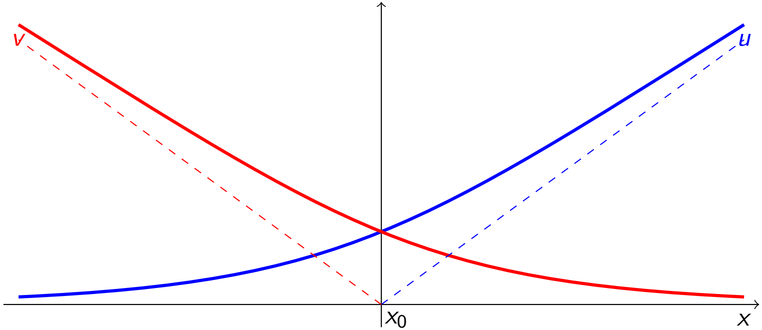
$$\begin{cases} u'' = uv^2 \\ v'' = vu^2 \, , \\ |u'|^2 + |v'|^2 - u^2 v^2 = h \\ u(x) = v(-x) \, , \qquad u, v > 0 \quad \text{in} \quad \mathbb{R}. \end{cases}$$

All these solution have linear growth at infinity

$$\lim_{|x| \to +\infty} \frac{u(x) + v(x)}{|x|} = \sqrt{h}$$

Selected solutions

One can seek particular solutions, having the reflectional symmetry: $v(x_0 - x) = u(x_0 + x)$, $\forall x \in \mathbb{R}$:



Theorem (H. Berestycki, T. C. Lin, J. Wei and C. Zhao, 2013, H. Berestycki, S. T., K. Wang, J. Wei, 2013)

Up to translations and reflections, there is only one one-parameter family of solutions to

$$\begin{cases} u'' = uv^2, \\ v'' = vu^2, \\ u, v > 0 \quad \textit{in } \mathbb{R} \end{cases}$$

For this family we have $v(x_0 - x) = u(x_0 + x), \forall x \in \mathbb{R}$. *In addition they are all stable.*

Monotone solutions in two dimensions

Theorem (H. Berestycki, T. C. Lin, J. Wei and C. Zhao (H. Berestycki, S. T., K.Wang, J. Wei, 2013))

Let (u, v) a solution to

$$\begin{cases} \Delta u = uv^2 \\ \Delta v = vu^2 \\ u, v > 0 \quad \text{in} \quad \mathbb{R}^2 \end{cases}$$

such that

$$u(x) + v(x) \leq C(1 + |x|).$$

and which is monotone in one direction. Then (u, v) is one dimensional , (i.e., there exists $a \in \mathbb{R}^2, |a| = 1, b \in \mathbb{R}$ such that $(u, v) = (u_0(a \cdot x - b), v_0(a \cdot x - b))$ where (u_0, v_0) is the one-dimensional solution).

Here monotone means that u and v are both monotone (in opposite ways) in one direction.

Stable solutions in two dimensions

A quite standard argument shows that monotone $\Longrightarrow$ stable. Recall that a stable solution (u, v) is such that the linearization is weakly positive definite. That is, it satisfies

$$\int_{\mathbb{R}^n} [|\nabla\varphi|^2 + |\nabla\psi|^2 + v^2\varphi^2 + u^2\psi^2 + 4uv\varphi\psi] \geq 0, \qquad \forall\varphi,\psi \in C_0^\infty(\mathbb{R}^n).$$

Theorem (H. Berestycki, S. T., K. Wang, J. Wei, 2013)

In two space dimensions, the previous theorem holds whenever monotone is replaced by stable.

Stronger results in any dimension: two theorems by Kelei Wang

Theorem (K. Wang, 2013)

In any space dimensions, let (u, v) a solution having at most linear growth and which is minimal (in the sense of Morse): the energy is minimized with respect to compact support variations. Then (u, v) is one dimensional .

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More problems: Gibbons type conjecture

Theorem (A. Farina, N. Soave 2013)

In any space dimensions, let (u, v) a solution having at most an algebraic growth, such that

$$\lim_{x_N \to -\infty} u(x', x_N) = 0; \qquad \lim_{x_N \to +\infty} u(x', x_N) = +\infty$$

$$\lim_{x_N \to -\infty} v(x', x_N) = +\infty; \qquad \lim_{x_N \to +\infty} v(x', x_N) = 0\ ,$$

the limits being uniform in the x'-variable. Then (u, v) is one dimensional .

Natural questions:

- do all entire solutions to the system have a linear growth?
- are there solutions with any power growth?
- are there solutions with exponential growth? Yes, N. Soave and A. Zilio, 2014.

Solutions with polynomial growth on the plane

For every integer d, we can build solutions to the system with polynomial growth $|x|^d$. To describe the behavior at infinity, let us consider the harmonic polynomial Φ of degree d as

$$\Phi := \mathrm{Re}(z^d).$$

Note that Φ has some dihedral symmetry; indeed, let us take its d nodal lines $L_1, \cdots, L_d$ and denote the corresponding reflection with respect to these lines as $T_1, \cdots, T_d$: then there holds $\Phi(T_i z) = -\Phi(z)$.

Theorem (H. Berestycki, S. T., K. Wang, J. Wei, 2013)

For each positive integer $d \geq 1$, there exists a solution (u, v) to the system, satisfying

1. $u - v > 0$ *in* $\{\Phi > 0\}$ *and* $u - v < 0$ *in* $\{\Phi < 0\}$;
2. $u \geq \Phi^+$ *and* $v \geq \Phi^-$;
3. $\forall i = 1, \cdots, d$, $u(T_i z) = v(z)$;

For a system with k components, define, for $r > 0$

$$N(r) := \frac{r \int_{B_r(0)} \sum_i |\nabla u_i|^2 + \sum_{j \neq i} u_i^2 u_j^2}{\int_{\partial B_r(0)} \sum_i u_i^2}$$

Then N is increasing (Almgren's monotonicity formula). Moreover, the solutions of the previous theorem do satisfy

$$\lim_{r \to +\infty} N(r) = d\ .$$

It can be shown that, for any nontrivial solution, of the system, there holds

$$\lim_{r \to +\infty} N(r) \geq 1\ .$$

Algebraic and linear growth

Definition (Growth rate)

We say that $(u_1, \ldots, u_k)$ has growth rate d if

$$\lim_{r \to +\infty} \frac{\frac{1}{r^{N-1}} \int_{\partial B_r} \sum_{i=1}^k u_i^2}{r^{2d'}} = \begin{cases} +\infty & \text{if } d' < d \\ 0 & \text{if } d' > d, \end{cases}$$

where B_r denotes the ball of center 0 and radius r.

We will show that any solution of (Sys(k)) has a growth rate, possibly infinite. We say that $(u_1, \ldots, u_k)$ has algebraic growth if it satisfies condition

$$u_1(x) + \cdots + u_k(x) \leq C(1 + |x|^d) \qquad \text{for every } x \in \mathbb{R}^N. \qquad \text{(AG(d))}$$

for some $C > 0$ and $d > 1$. If the above condition holds with $d = 1$, we say that $(u_1, \ldots, u_k)$ has linear growth.

Asymptotics at infinity

existence of the renormalized limiting profiles (blow-down)

We consider the rescaled sequence

$$u_R(x), v_R(x)) := (\tfrac{1}{L(R)}u(Rx)\tfrac{1}{L(R)}v(Rx)),$$

where $u(0) = v(0)$ and $L(R)$ is chosen so that

$$\int_{\partial B_1(0)} u_R^2 + v_R^2 = 1.$$

We have the following

Theorem

Let (u, v) be a solution of the system such that $d := \lim_{r\to+\infty} N(r) < +\infty$. As $R \to \infty$, (u_R, v_R) defined above (up to a subsequence) converges to (Ψ^+, Ψ^-) uniformly on any compact set of $\mathbb{R}^N$, where Ψ is a homogeneous harmonic polynomial of degree d.

Theorem (H. Berestycki, S.T., K. Wang, J. Wei 2013)

For every integers h and k, there exists a positive solution to the system Sys(k) on $\mathbb{C} \simeq \mathbb{R}^2$ having the following symmetries (here $\overline{z}$ is the complex conjugate of z):

$$\begin{aligned} u_i(z) &= u_i(G^h z), & &\text{on } \mathbb{C}, i = 1, \dots, k \\ u_i(z) &= u_{i+1}(Gz), & &\text{on } \mathbb{C}, i = 1, \dots, k \\ u_{k+1}(z) &= u_1(z), & &\text{on } \mathbb{C} \\ u_{k+2-i}(z) &= u_i(\overline{z}), & &\text{on } \mathbb{C}, i = 1, \dots, k \end{aligned}$$

such that

$$\lim_{r \to +\infty} \frac{r \int_{B_r(0)} \sum_1^k |\nabla u_i|^2 + \sum_{i<j} u_i^2 u_j^2}{\int_{\partial B_r(0)} \sum_1^k u_i^2} = d := \frac{hk}{2} .$$

Note that $2d$ is the order of the symmetry group acting on the function space.

Remarks

- The blow-down Theorem allows us to associate to any positive solution (u, v) of (Sys(2)) a homogeneous harmonic polynomial as its limiting profile. This implies a quantization of the admissible growth rates at infinity. Indeed, if $N = 2$, d is a half-integer.

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- Though the limiting profiles of renormalized rescalements still exist, understanding the quantization of d becomes far more complicated in higher dimension.
- Our existence Theorem ensures the existence of a positive solution to (Sys(k)) with minimal growth rate $3/2$ when $k = 3$, 2 when $k = 4$, $5/2$ when $k = 5$, ... We wonder if these are really the minimal admissible growth rates or not.

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- Though the limiting profiles of renormalized rescalements still exist, understanding the quantization of d becomes far more complicated in higher dimension.
- Our existence Theorem ensures the existence of a positive solution to (Sys(k)) with minimal growth rate $3/2$ when $k = 3$, 2 when $k = 4$, $5/2$ when $k = 5$, ... We wonder if these are really the minimal admissible growth rates or not.
- Conversely, is it true that if a nonnegative solution of (Sys(k)) has growth rate d, then there exists a maximal number of nontrivial components depending on d and, possibly, on the dimension N?

Spectral minimal partitions

Definition

Let $1 \leq k \in \mathbb{N}$. A *k-partition* (or, simply, *partition*) of $\mathbb{S}^{N-1}$ is a family $\omega = (\omega_1, \ldots, \omega_k)$ of mutually disjoint open and connected subsets $\omega_i \subset \mathbb{S}^{N-1}$. We denote the class of the k-partition of $\mathbb{S}^{N-1}$ as $\mathcal{P}_k(\mathbb{S}^{N-1})$.

Consider the spectral optimal partition sequence ($k \in \mathbb{N}$):

$$\mathfrak{L}_k(\mathbb{S}^{N-1}) = \inf_{\omega \in \mathcal{P}_k} \max_i \lambda_1(\omega_i)$$

where $\mathcal{P}_k$ is the set of open connected k-partitions of $\mathbb{S}^{N-1}$ and $\lambda_1(\omega)$ denotes the first eigenvalue of the Laplace-Beltrami operator with Dirichlet boundary conditions on $\partial\omega \subset S^{N-1}$.
If $k = 2$ then $\mathfrak{L}_2 = \lambda_2(\mathbb{S}^{N-1})$ and the optimal 2-partition is the nodal set of φ_2: two half spheres. If $N = 2$, it is easy to prove that, for every k

$$\mathfrak{L}_k(\mathbb{S}^1) = \frac{k^2}{4}\ .$$

In general, the exact value of $\mathfrak{L}_k(\mathbb{S}^{N-1})$ is not known.

Courant sharpness and deficiency

In some few cases, in order to compute $\mathfrak{L}_k(\mathbb{S}^{N-1})$, one can look at the nodal partition associated with an eigenvalue.

Theorem (B. Helffer, T. Hoffmann-Ostenhof, S. T. Ann. IHP 2009)

If the graph of a minimal partition is bipartite, then it is the nodal domain of an eigenfunction φ_j*.*

Theorem (B. Helffer, T. Hoffmann-Ostenhof, S. T. (2009-10))

The k*–th eigenfunction has exactly* k *nodal domains (i.e. is sharp with respect to the Courant nodal Theorem) if and only if the associated nodal* k*–partition is optimal.*

G. Berkolaiko, P. Kuchment and U. Smilanski (2012) proved that *generically* the deficiency of nodal domains of the k-the eigenfunction is equal to the Morse index (in a suitable definition) of the associated partition, withe respect to the cost function of the minimal partition problem.

A digression on the Bishop conjecture

We consider the Laplace-Beltrami operator on the two-sphere.

Bishop conjecture, 1992

The minimal 3-partition for the arithmetic average of the eigenvalues:

$$\frac{1}{3}\left(\sum_{i=1}^{3} \lambda_1(D_i)\right)$$

corresponds to the **Y**-partition, whose boundary is given by the intersection of $\mathbb{S}^2$ with the three half-planes defined respectively by $\phi = 0, \frac{2\pi}{3}, \frac{-2\pi}{3}$

Bishop's Conjecture was motivated by the analysis of the properties of harmonic functions in conic sets. A reference paper in this context is that by Friedland-Hayman (Commentarii Mathematici Helvetici, 1976). It is proved there the nontrivial fact that the optimal two-partition is achieved by the two half spheres.

Uniqueness for $\mathfrak{L}_3$ in the two-sphere

Theorem (B. Helffer, T. Hoffman-Ostenhof, S.T, 2010)

Any minimal spectral 3-partition of $\mathbb{S}^2$ is (up to a rotation) obtained by the **Y***-partition. Hence*

$$\mathfrak{L}_3(\mathbb{S}^2) = \frac{15}{4}\,.$$

Consider a locally Lipschitz continuous α-homogeneous function in $\mathbb{R}^3$, that is of the form:

$$u(x) = r^\alpha g(\theta, \phi)$$

which is harmonic outside its nodal set: $-\Delta u = 0$ when $u > 0$ and such that the nodal set divides the sphere in (at least) three parts, then

$$\alpha(\alpha+1) \geq \mathfrak{L}_3(\mathbb{S}^2)\,.$$

Hence our theorem implies that $\alpha \geq 3/2$.

Papers

- N. Soave and S. Terracini *Liouville theorems and 1-dimensional symmetry for solutions of an elliptic system modeling phase separation*, preprint 2014 (arXiv:1404.7288)
- M. Ramos, H. Tavares and S. Terracini, *Existence and regularity of solutions to optimal partition problems involving Laplacian eigenvalues*, preprint 2014 (arXiv:1403.6313)

Minimal growth rate for competition-diffusion systems having algebraic growth

Theorem

Let $N \geq 2$, and let $(u_1, \dots, u_k)$ be a positive solution of (Sys(k)) *having algebraic growth, that is, there exists $C, d > 0$ such that*

$$u_1(x) + \cdots + u_k(x) \leq C(1 + |x|^d) \qquad \text{for every } x \in \mathbb{R}^N. \tag{1}$$

Then

$$d \geq \sqrt{\left(\frac{N-2}{2}\right)^2 + \mathfrak{L}_k(\mathbb{S}^{N-1})} - \frac{N-2}{2} := \gamma(\mathfrak{L}_k(\mathbb{S}^{N-1})),$$

where $\mathfrak{L}_k(\mathbb{S}^{N-1})$ is the spectral minimal partition sequence of $-\Delta_{\mathbb{S}^{N-1}}$. Note that $\mathfrak{L}_k(\mathbb{S}^1) = k^2/4$ when $N = 2$.

In one direction, this means that if we consider a positive solution of the system with a prescribed number k of components, then we have a minimal admissible growth for the solution itself.

The minimal growth is strictly increasing in k. On the other hand a bound on the growth of a positive solution imposes a bound on the number of components k of the solution itself.

Minimal growth estimates

The minimal growth is strictly increasing in k. On the other hand a bound on the growth of a positive solution imposes a bound on the number of components k of the solution itself.

As said, when $N \geq 3$, the exact value of $\mathfrak{L}_k(\mathbb{S}^{N-1})$ is known for $k = 2$, and $k = 3$ and $N = 3$. We first extend the result by Helffer, Hoffman-Ostenhof and T. to the sphere of any dimension.

Theorem

A minimal spectral 3-partition of $\mathbb{S}^{N-1}$ is (up to a rotation) obtained by the **Y***-partition. Hence*

$$\mathfrak{L}_3(\mathbb{S}^{N-1}) = \frac{3(N+2)}{4}.$$

The low growth theorem

As a consequence, we will prove the following.

Theorem

Let $N, k \geq 2$, and let $(u_1, \dots, u_k)$ be a positive solution of (Sys(k))*.*
(i) If the solution has linear growth, that is there exists $C > 0$ such that

$$u_1(x) + \cdots + u_k(x) \leq C(1 + |x|) \qquad \text{for every } x \in \mathbb{R}^N,$$

then $k = 2$ and the solution has growth rate 1.
(ii) If there exists $C > 0$ such that

$$u_1(x) + \cdots + u_k(x) \leq C(1 + |x|^{3/2}) \qquad \text{for every } x \in \mathbb{R}^N.$$

Then either $k = 2$ and the solution has linear growth, or $k = 3$ and the solution has growth rate $3/2$.

Two characterizations of the linear growth case.

Theorem (S-T)

Let $N \geq 2$, let $(u_1, \dots, u_k)$ be a nonnegative solution of (Sys(k)).

(i) *If $(u_1, \dots, u_k)$ has linear growth, then all the components but two, say u_1 and u_2, are identically zero, and (u_1, u_2) is 1-dimensional.*

(ii) *If $(u_1, \dots, u_k)$ has algebraic growth and for some $i \neq j$*

$$\lim_{x_N \to \pm\infty} (u_i(x', x_N) - u_j(x', x_N)) = \pm\infty,$$

the limits being uniform in $x' \in \mathbb{R}^{N-1}$, then $(u_1, \dots, u_k)$ has linear growth, all the components u_l with $l \neq i, j$ are identically zero, and (u_i, u_j) is 1-dimensional.

Rescaling and limiting profiles for many components systems

The first of our main results is the extension of the blow-down Theorem in the case of many components.

Theorem

Let $N, k \geq 2$, *let* $\mathbf{u}$ *be a nonnegative solution of the system, and assume*

$$\lim_{r \to +\infty} N(\mathbf{u}, 0, r) =: d < +\infty.$$

Then, up to a subsequence,

$$\mathbf{u}_R \to \mathbf{u}_\infty = r^d (g_1(\theta), \ldots, g_k(\theta)) \quad \text{as } R \to +\infty$$

in $\mathcal{C}^0_{loc}(\mathbb{R}^N)$ *and in* $H^1_{loc}(\mathbb{R}^N)$, *where* $(r, \theta) \in [0, +\infty) \times \mathbb{S}^{N-1}$ *is a system of polar coordinates in* $\mathbb{R}^N$ *centred in* 0.

Furthermore:

- the components $u_{i,\infty}$ are nonnegative and with disjoint support: $u_{i,\infty}u_{j,\infty} \equiv 0$ for every $i \neq j$;
- $\Delta u_{i,\infty} = 0$ in the positivity domain $\{u_{i,\infty} > 0\}$;
- if for some $i \neq j$ there exists two adjacent nodal domains $B_i \subset \{u_{i,\infty} > 0\}$ and $B_j \subset \{u_{j,\infty} > 0\}$, then $u_{i,\infty} - u_{j,\infty}$ is harmonic in $\mathrm{Int}(\overline{B_i \cup B_j})$;
- the set $\{\mathbf{u}_\infty = \mathbf{0}\} \cap \partial B_1$ has null $(N-1)$-dimensional measure;
- $H(\mathbf{u}, 0, R)R^2 \sum_{i<j} u_{i,R}^2 u_{j,R}^2 \to 0$ in $L^1_{loc}(\mathbb{R}^N)$.

Furthermore, when $N = 2\ d$ is a half-integer. Moreover, letting

$$\Psi_d(r,\theta) := \frac{1}{\sqrt{\pi}} r^d \sin(d\theta),$$

there exists a partition $(A_1, \ldots, A_k)$ of the positivity domain $\Sigma_{|\Psi_d|} = \{|\Psi_d| > 0\}$, where for every i A_i is the union of non-adjacent nodal domains of $\Sigma_{|\Psi_d|}$, such that, up to a subsequence and up to a rotation, $\mathbf{u}_R \to (\chi_{A_1}, \ldots, \chi_{A_k})|\Psi_d|$ as $R \to +\infty$, in $\mathcal{C}^0_{loc}(\mathbb{R}^N)$ and in $H^1_{loc}(\mathbb{R}^N)$.

Remarks

1) The same result holds for blow-down sequences centered at any point $x_0 \neq 0$.
2) By the Almgren's monotonicity formula, it follows that the limit $N(\mathbf{u}, 0, +\infty)$ always exists. Moreover, it is finite if and only if $\mathbf{u}$ has algebraic growth.

In two space dimension, when k is odd we have to take into account the possibility that the homogeneity degree of the limiting profile is half integer. Hence, when k is odd, Ψ_d does not define a harmonic function in $\mathbb{R}^2$, since it is not 2π-periodic in the argument θ; it can be seen as a harmonic function in the double covering $\{r \geq 0, 0 \leq \theta < 4\pi\}$.

In higher dimensions, the structure of the limiting profile is far more complicated, (cfr, H. Tavares and S.T. 2013)

Clean up lemma

Proposition

Let $\mathbf{u}$ *be a nonnegative solution of* (Sys(k)) *having algebraic growth. Then* $d := N(\mathbf{u}, 0, +\infty) \in (0, +\infty)$ *is the growth rate of* $\mathbf{u}$*.*

Hence, the value of d characterizes the maximal number of non-trivial components for a limiting profile, which hopefully should coincide with the maximal number of non-trivial components of the "original" solution.

Lemma

Let $N \geq 2$*, and let* $\mathbf{u}$ *be a nonnegative solution of* (Sys(k)) *having algebraic growth. Let us assume that there exists a sequence* $R_n \to +\infty$ *as* $n \to \infty$*, such that* $u_{i,R_n} \to 0$ *in* $\mathcal{C}^0_{\mathrm{loc}}(\mathbb{R}^N)$ *for some* i*. Then* $u_i \equiv 0$ *in* $\mathbb{R}^N$*.*

Liouville type theorem in dimension two

Next we prove the Liouville-type Theorem in dimension 2. In terms of the Almgren frequency function, it can be re-phrased as follows.

Theorem

Let $N = 2$, $k \geq 2$, and let $\mathbf{u} = (u_1, \ldots, u_k)$ be a nonnegative solution of (Sys(k)) *such that $N(\mathbf{u}, 0, +\infty) =: d \in (0, +\infty)$. Then at most 2d components of $\mathbf{u}$ do not vanish identically.*

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Equivalently, let $N = 2$, $k \geq 2$, and let $\mathbf{u} = (u_1, \ldots, u_k)$ be a positive solution of (Sys(k)) such that $N(\mathbf{u}, 0, +\infty) =: d \in (0, +\infty)$. Then $d \geq k/2$.

Liouville type theorem in higher dimensions

We recall the definitions of optimal spectral partition and that of characteristic exponent

$$\mathcal{L}_k(\mathbb{S}^{N-1}) := \inf_{\omega \in \mathcal{P}_k(\mathbb{S}^{N-1})} \max_{i=1,\dots,k} \lambda_1(\omega_i)$$

$$\gamma(t) := \sqrt{\left(\frac{N-2}{2}\right)^2 + t} - \left(\frac{N-2}{2}\right).$$

Theorem

Let $N, k \geq 2$, and let $\mathbf{u} = (u_1, \dots, u_k)$ be a nonnegative solution of (Sys(k)) such that $N(\mathbf{u}, 0, +\infty) =: d \in (0, +\infty)$. If m is the maximal positive integer such that $\gamma(\mathcal{L}_m(\mathbb{S}^{N-1})) \leq d$, then at most m components of $\mathbf{u}$ do not vanish identically.
Equivalently, let $N, k \geq 2$, and let $\mathbf{u}$ be a positive solution of (Sys(k)) such that $N(\mathbf{u}, 0, +\infty) =: d \in (0, +\infty)$. Then $d \geq \gamma(\mathcal{L}_k(\mathbb{S}^{N-1}))$.

This previous statement is the key of the proof of the Low Growth Theorem ($d \leq 3/2$). The following re-formulation is more suited to describe our result.

Corollary

Let $N, k \geq 2$, and let $\mathbf{u} = (u_1, \ldots, u_k)$ be a nonnegative solution of (Sys(k)) *such that $N(\mathbf{u}, 0, +\infty) =: d \in (0, +\infty)$. Then either $d = 1$ or $d \geq 3/2$. Furthermore:*

- *(i) if $d = 1$, then $\mathbf{u}$ has exactly 2 non-trivial components (and is a one dimensional solution);*
- *(ii) if $d = 3/2$, then $\mathbf{u}$ has exactly 3 non-trivial components.*

Problems

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- Uniqueness of the **Y**-partition as the optimal 3-partition, up to rotations, in any dimension.
- Uniqueness of the planar Berestycki-Lin-T.-Wei solutions of Sys(k) and of Sys(2) for any k, h in the planar case.

Problems

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