Harmonic approximation and improvement of flatness in singular perturbation problems

Kelei Wang

Wuhan Institute of Physics and Mathematics
Chinese Academy of Sciences

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Let $u \in C^2(\mathbb{R}^n)$ be a solution of

$$\Delta u = u^3 - u$$

such that $|u| < 1$ and $\frac{\partial u}{\partial x_n} > 0$ in $\mathbb{R}^n$. Is it true that u is one dimensional at least if $n \leq 8$?

Ghoussoub-Gui, Ambrosio-Cabre, Savin, Del Pino-Kowalczyk-Wei etc.

Asymptotical flatness $\Longrightarrow$ Flatness

Rescaling $u(x) \mapsto u(\varepsilon^{-1}x) \Longrightarrow$ asymptotic behavior of the level sets of u.

$$\varepsilon^2 \Delta u_\varepsilon = u_\varepsilon - u_\varepsilon^3.$$

$\{u_\varepsilon = 0\} \to \Sigma$ a minimal hypersurface.
Monotonicity of $u \Longrightarrow \varepsilon\{u = 0\}$ converges to a minimal graph $\Longrightarrow$ a hyperplane by the Bernstein theorem, if $n \leq 8$.

$$\text{Asymptotical flatness of } \{u = 0\} \Longrightarrow \{u = 0\} \text{ flat.}$$

Let Ω be a set with bounded perimeter ($\Longleftrightarrow \chi_\Omega \in BV$). Assume $\partial\Omega$ has minimal perimeter, in the sense that

$$\|\partial\Omega \cap B_R\| \leq \|\partial\Omega' \cap B_R\|,$$

for any Ω' satisfying $\Omega' = \Omega$ outside B_R.

Theorem (De Giorgi 1960)

There exists an $\varepsilon(n) > 0$, if $\partial\Omega$ is minimal and $\partial\Omega \cap B_1 \subset \{|x_{n+1}| \leq \varepsilon(n)\}$, then $\partial\Omega \cap B_{1/2}$ is a smooth hypersurface.

Later developed by Almgren, Allard and many others.

The excess

$$E(\Sigma; r, \mathbb{R}^n) := r^{-n} \int_{B_r^n \times (-1,1)} \left[1 - (\nu \cdot e_{n+1})^2\right] d\|\Sigma\|.$$

1. ν: the (weak) normal vector of $\partial\Omega$, which exists a.e..
2. $\left[1 - (\nu \cdot e_{n+1})^2\right] \sim |\nu - e_{n+1}|^2$.
3. If $\Sigma = \text{graph}(h)$,

$$\int_{B_1^n \times (-1,1)} \left[1 - (\nu \cdot e_{n+1})^2\right] d\|\Sigma\| = \int_{B_1^n} \frac{|\nabla h|^2}{1 + |\nabla h|^2}.$$

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Improvement of tilt-excess decay

Theorem

There exist universal constants ε, θ (small), if $E(\Sigma; 1, \mathbb{R}^n) \leq \varepsilon$, then we can find another plane T, satisfying

$$\|T - \mathbb{R}^n\| \leq CE(\Sigma; 1, \mathbb{R}^n)^{\frac{1}{2}},$$

such that

$$E(\Sigma; \theta, T) \leq \frac{1}{2} E(\Sigma; 1, \mathbb{R}^n).$$

Iterating the improvement of tilt-excess decay, we get a sequence of plane T_i such that

$$E(\theta^i; 0, T_i) \le C\varepsilon 2^{-i},$$

where $\|T_i - T_{i+1}\| \le C\varepsilon 2^{-i/2}$.

- T_i converges to a limit $T_\infty \Longrightarrow$ the tangent plane of Σ at 0.
- The Morrey type bound $E(\theta^i; 0, T_\infty) \le C\varepsilon 2^{-i} \Longrightarrow$ Hölder continuity of the tangent plane.

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Harmonic approximation

If $\Sigma = \text{graph}(h)$,

$$\int_{B_1^n \times (-1,1)} \left[1 - (\nu \cdot e_{n+1})^2\right] d\|\Sigma\| = \int_{B_1^n} \frac{|\nabla h|^2}{1 + |\nabla h|^2} \approx \int_{B_1^n} |\nabla h|^2.$$

1. **Lip approximation:** $\partial\Omega$ lies on the graph of a Lipschitz function h, except a set of small measure.
2. By choosing special test functions (normal variation) in **the stationary condition** for Σ,

$$\int \text{div} X d\Sigma = 0,$$

$\Longrightarrow E^{-1/2} h$ almost harmonic.

Harmonic approximation

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$$\begin{aligned}\int_{B_1^n\times(-1,1)}\left[1-(\nu\cdot e_{n+1})^2\right]d\|\Sigma\| &= \int_{B_1^n}\frac{|\nabla h|^2}{1+|\nabla h|^2}\\ &\approx \int_{B_1^n}|\nabla h|^2.\end{aligned}$$

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Basic decay estimates for harmonic functions

If h is a harmonic function satisfying

$$\int_{B_1^n} |\nabla h|^2 \leq 1,$$

then

$$r^{-n} \int_{B_r^n} |\nabla h - \nabla h(0)|^2 \leq C(n) r^2 < \frac{1}{2}$$

for $r \leq \theta(n)$.

The singularly perturbed Allen-Cahn equation

$$\varepsilon \Delta u_\varepsilon = \frac{1}{\varepsilon} W'(u_\varepsilon). \tag{1}$$

W is a standard double well potential.
Example: $W(u) = \left(1 - u^2\right)^2 / 4$.
Question: if $\{u_\varepsilon = 0\} \cap B_1 \subset \{|x_n| \le h\}$ with h sufficiently small, is it a smooth hypersurface?

Excess

Definition

Let P be an n-dimensional hyperplane in $\mathbb{R}^{n+1}$ and e one of its unit normal vector, $B_r(x) \subset P$ an open ball and $\mathcal{C}_r(x) = B_r(x) \times (-1, 1)$ the cylinder over $B_r(x)$. The excess of u_ε in $\mathcal{C}_r(x)$ with respect to P is

$$E(r; x, u_\varepsilon, P) := r^{-n} \int_{\mathcal{C}_r(x)} \left[1 - (\nu_\varepsilon \cdot e)^2\right] \varepsilon |\nabla u_\varepsilon|^2 dX. \qquad (2)$$

Here $\nu_\varepsilon = \nabla u_\varepsilon / |\nabla u_\varepsilon|$.
If $\{u_\varepsilon = t\} = \{x_{n+1} = h(x, t)\}$, then the excess has the form

$$\int_{-1}^{1} \left(\int \frac{|\nabla h(x,t)|^2}{1 + |\nabla h(x,t)|^2} \varepsilon |\nabla u_\varepsilon(x, h(x,t))| dx \right) dt,$$

Tilt-Excess Decay

Theorem (Tilt-excess decay)

$\exists\delta_0, \tau_0, \varepsilon_0 > 0$, $\theta \in (0, 1/4)$ and K_0 large so that the following holds. Let u_ε be a solution of (1) with $\varepsilon \le \varepsilon_0$ in $\mathcal{B}_4$, satisfying the Modica inequality, $|u_\varepsilon(0)| \le \gamma$, and

$$4^{-n}\int_{\mathcal{B}_4} \frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon) \le (1+\tau_0)\,\sigma_0\omega_n, \tag{3}$$

$$\delta_\varepsilon^2 := E(2; 0, u_\varepsilon, \mathbb{R}^n) \le \delta_0^2, \tag{4}$$

where $\delta_\varepsilon \ge K_0\varepsilon$. Then there exists another plane P, such that

$$\theta^{-n}E(\theta; 0, u_\varepsilon, P) \le \frac{\theta}{2}E(2; 0, u_\varepsilon, \mathbb{R}^n), \tag{5}$$

$$\|e - e_{n+1}\| \le CE(2; 0, u_\varepsilon, \mathbb{R}^n)^{1/2},$$

where e is the unit normal vector of P pointing to the above.

The stationary condition

For any smooth vector field Y with compact support, by considering the domain variation in the form

$$u_\varepsilon^t(X) := u_\varepsilon(X + tY(X)), \quad \text{for } |t| \text{ small},$$

from the definition of critical points we get

$$\frac{d}{dt}\int \frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon)\Big|_{t=0} = 0.$$

Then after some integration by parts, we obtain the stationary condition satisfied by u_ε:

$$\int \left(\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon)\right) \operatorname{div} Y - \varepsilon DY(\nabla u_\varepsilon, \nabla u_\varepsilon) = 0.$$

Lipschitz approximation

- For each $t \in (-1,1)$, $\{u_\varepsilon = t\}$ lies on the graph of Lipschitz functions $x_{n+1} = h_\varepsilon^t(x)$, except a set of small $\mathcal{H}^n$ measure.
- $\frac{\partial h_\varepsilon^t}{\partial t} \sim \varepsilon \Longleftrightarrow$ the width of transition layer $\sim \varepsilon$.
- $\delta_\varepsilon \gg \varepsilon \Longrightarrow \delta_\varepsilon^{-1} h_\varepsilon^t$ converges to the same limit, $\forall t \in (-1,1)$.

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Proof of the tilt-excess decay

1. $\forall b > 0, \int_{-1+b}^{1-b} \int_{B_1^n} |\nabla h_\varepsilon^t|^2 dxdt \leq C(b)\delta_\varepsilon^2$;
2. $h := \lim \delta_\varepsilon^{-1} h_\varepsilon^t$ is harmonic, by using special test functions in the stationary condition of u_ε;
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The condition $\delta_\varepsilon \gg \varepsilon$ is violated after a finite times of iterating the tilt-excess decay estimate.

Lemma

There exist two universal constants K_1 *and* K_2, *such that for any* $X_0 \in \{|u_\varepsilon| \leq \gamma\} \cap \mathcal{B}_1$ *and ball* $\mathcal{B}_r(X_0)$ *with* $r \in (K_1\varepsilon, \theta)$, *we can find a unit vector* $e_r(X_0)$ *to satisfy*

$$r^{-n} \int_{\mathcal{B}_r(X_0)} \left[1 - (\nu_\varepsilon \cdot e_r(X_0))^2\right] \varepsilon |\nabla u_\varepsilon|^2 \leq \max\{K_2\varepsilon^2 r^{-2}, K_2\tau_A^2 r^\alpha\}. \tag{6}$$

Here $\alpha = \frac{\log 2}{|\log \theta|} \in (0, 1)$.

A dichotomy phenomena: if at scale θ^i, $\delta_{\varepsilon_i} \gg \varepsilon_i$, then this condition holds for all $j \leq i$.

A direct proof of De Giorgi conjecture for minimizers

Let u be a minimizer of

$$\int \frac{|\nabla u|^2}{2} + W(u)$$

in $\mathbb{R}^{n+1}$.

1. **Energy bound** $\int_{B_R} \frac{|\nabla u|^2}{2} + W(u) \leq CR^n$;
2. $u_\varepsilon(x) := u(\varepsilon x) \to \chi_H - \chi_{H^c}$, where H is a minimal cone $\Longleftarrow$ the monotonicity formula;
3. If $n \leq 6$, H is an half space (by Simons). But it may depend on the blow down sequence.

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Using the tilt-excess decay we can prove

Lemma (Uniqueness of the blow down limit)

There exists a unit vector e_∞ and a universal constant $C(n)$ such that

$$\int_{\mathcal{B}_R} \left[1 - (\nu \cdot e_\infty)^2\right] |\nabla u|^2 \leq C(n) R^{n-2}, \quad \forall R > 1. \tag{7}$$

Then we can use the sliding method to show that u is indeed one dimensional.

Local version $\Longrightarrow$ the level sets of u are Lipschitz graphs in one fixed direction.

Final step: Lipschitz $\Longrightarrow$ $C^{1,\alpha}$ by Caffarelli and Cordoba.

Savin's improvement of flatness

Theorem (Savin)

If u_ε *is a minimizer in* B_1, $u_\varepsilon(0) = 0$ *and* $\{u_\varepsilon = 0\} \subset \{|x_{n+1}| < \delta\}$, *where* $\varepsilon \leq \varepsilon_0$ *and* $\delta \leq \delta_0$, *then*

$$\{u_\varepsilon = 0\} \cap B_{\eta_1} \subset \{|e \cdot x| \leq \eta_2 \delta\}.$$

Here $0 < \eta_2 < \eta_1 < 1$ *are universal,* $|e - e_{n+1}| \leq C\delta$.

Ingredients: harmonic approximation in L^∞ sense (viscosity solutions) and Harnack inequality from a Krylov-Safonov type argument.

Advantage: only the single level let $\{u = 0\}$.

An elliptic system

$$\Delta u = uv^2, \ \Delta v = vu^2, \ u, v > 0 \text{ in } \mathbb{R}^n. \tag{8}$$

Problem: Let (u, v) be a solution to (1) such that

$$u(x) + v(x) \leq C(1 + |x|),$$

is it one dimensional (i.e. depending only on one variable)?

The singularly perturbed problem

Theorem (Caffarelli-Lin, Tavares-Terracini, Dancer-W-Zhang)

$$\begin{cases} \Delta u_\kappa = \kappa u_\kappa v_\kappa^2, \\ \Delta v_\kappa = \kappa v_\kappa u_\kappa^2. \end{cases} \tag{9}$$

Assume that as $\kappa \to +\infty$, a sequence of solutions of (9), $(u_\kappa, v_\kappa) \to (u, v)$ uniformly, then $uv \equiv 0$ and $u - v$ is harmonic.

Solutions are critical points of the functional

$$E_\kappa = \int |\nabla u_\kappa|^2 + |\nabla v_\kappa|^2 + \kappa u_\kappa^2 v_\kappa^2.$$

Suitable rescalings of (u_κ, v_κ) converge to an entire solution of (1).

Blowing down

Let (u, v) be a solution of (1) with a linear growth. Define

$$(u_R(x), v_R(x)) := (\frac{1}{L(R)} u(Rx), \frac{1}{L(R)} v(Rx)),$$

where $L(R)$ is chosen so that $\int_{\partial B_1(0)} u_R^2 + v_R^2 = 1$.
(u_R, v_R) satisfies (9) with $\kappa = L(R)^2 R^2$.
Almgren monotonicity formula $\Longrightarrow$ As $R \to +\infty$, $(u_R, v_R) \to (x_1^+, x_1^-)$.

$$\frac{1}{R}\{u - v = t\} = \{u_R - v_R = \frac{t}{L(R)}\} \to \{x_1 = 0\}$$

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Asymptotically flat $\Rightarrow$ flat?

An important intermediate result for our analysis is

Theorem

Let (u, v) be a solution of (1) in $\mathbb{R}^n$ with linear growth at infinity, then (u, v) is globally Lipschitz continuous.

Consequences: $\int_{B_R} u^2 v^2 \sim R^{n-1}$.

Theorem

There exist four universal constants $\theta \in (0, 1/2)$, ε_0 *small and* K_0, $C(n)$ *large, if* (u_κ, v_κ) *is a solution of* (9) *in* $B_1(0)$, *satisfying*

$$\int_{B_1(0)} |\nabla u_\kappa - \nabla v_\kappa - e|^2 = \varepsilon^2 \leq \varepsilon_0^2, \tag{10}$$

where e *is a vector satisfying* $|e| \geq c_0/2$, *and* $\kappa^{1/4}\varepsilon^2 \geq K_0$, *then there exists another vector* $\tilde{e}$, *with*

$$|\tilde{e} - e| \leq C(n)\varepsilon,$$

such that

$$\theta^{-n} \int_{B_\theta(0)} |\nabla u_\kappa - \nabla v_\kappa - \tilde{e}|^2 \leq \frac{1}{2}\varepsilon^2.$$

Blow up the stationary condition

The stationary condition for (u_κ, v_κ),

$$\int_{B_1(0)} \left(|\nabla u_\kappa|^2 + |\nabla v_\kappa|^2 + \kappa u_\kappa^2 v_\kappa^2\right) \operatorname{div} Y - 2DY(\nabla u_\kappa, \nabla u_\kappa) \\ -2DY(\nabla v_\kappa, \nabla v_\kappa) = 0.$$

We can use this condition to show that

$$\bar{w}_\kappa := \frac{1}{\varepsilon_\kappa}\left(u_\kappa - v_\kappa - e \cdot x\right),$$

converges to a harmonic function.
Combining a Caccioppoli type inequality with the stationary condition using a special deformation in the normal direction, we can prove $\bar{w}_\kappa$ converges strongly in H^1.

Uniqueness of the blow down limit

By an standard iteration using the improvement of flatness result we can prove

Theorem

There exists a constant $C > 0$ such that, for any $x \in \{u = v\}$ and $R > 1$, there exists a vector $e_{x,R}$, with

$$|e_{x,R}| \geq c_0/2,$$

satisfying

$$\int_{B_R(x)} |\nabla u - \nabla v - e_{x,R}|^2 \leq CR^{n-1}.$$

This theorem implies the uniqueness of the blowing down limit$\Longrightarrow$ the one dimensional result by a sliding method.

Thanks !