Nonlinear elliptic systems with mixed couplings

Zhi-Qiang Wang

Utah State / CIM Nankai

Banff, May 27, 2014

Coupled nonlinear elliptic equations

- 2-system

$$\begin{cases} -\Delta u + \lambda_1 u = \mu_1 u^3 + \beta v^2 u, \text{in } \Omega \\ -\Delta v + \lambda_2 v = \mu_2 v^3 + \beta u^2 v, \text{in } \Omega \\ u = v = 0, \text{ on } \partial\Omega \end{cases} \tag{1}$$

Here $\Omega \subset \mathbb{R}^n$ a domain, bounded or unbounded, $n \leq 3$.

- general $N-$system

$$-\Delta u_j + \lambda_j u_j = \mu_j u_j^3 + \sum_{k=1, k\neq j}^{N} \beta_{jk} u_k^2 u_j, \tag{2}$$

$j = 1, ..., N$. $\lambda_j, \mu_j, \beta_{kj} = \beta_{jk}$ real numbers. Writing $\mu_j = \beta_{jj}$, the N-system

$$-\Delta u_j + \lambda_j u_j = \sum_{k=1}^{N} \beta_{jk} u_k^2 u_j, \ j = 1, ..., N$$

- Applications: Models for Bose Einstein condensates (BEC), standing waves of the time-dependent nonlinear Schrödinger systems.
- $\beta_{jk}, (j \neq k)$: (nonlinear) coupling constants whose sizes and signs play an important role on the existence, nonexistence, multiplicity, and qualitative property of solutions, mathematically providing new features and phenomena comparing with scalar equations

Synchronization vs Segregation

- $\beta_{jk} > 0$, attractive case – components of solutions tend to be spatially **synchronized**;
  $\beta_{jk} < 0$, repulsive case – components of solutions tend to be spatially **segregated**
- we focus on **positive vector solutions** with synchronization and segregation phenomena

- **semi-trivial solutions**,

which are zero functions for at least one component.

For example, for equation (1), $(w_1, 0)$ and $(0, w_2)$ are solutions of (1) if w_j solves

$$-\Delta w_j + \lambda_j w_j = \mu_j w_j^3$$

In general, the system collapses to lower order sub-systems

- **Non existence of positive solutions**

For example, for $N = 2$, if $\lambda_1 \geq \lambda_2$ and $\mu_1 < \mu_2$, then there is no positive solution for

$$\beta = \beta_{12} \in [\mu_1, \mu_2]$$

The attractive case

- **Synchronized positive solutions**

Let $\lambda_1 = \cdots \lambda_N = 1$, $\mu_j > 0$. Then (2) has a positive solution of the form

$$u_j(x) = \alpha_j w(x),\ j = 1, \dots, N,$$

where w is the unique radial positive solution of

$$-\Delta w + w = w^3,$$

if the matrix $B = (\beta_{ij})$ satisfies an algebraic condition (e.g. T. Bartsch-Wang, 06).

For $N = 2$,

$$\beta \in (-\sqrt{\mu_1 \mu_2}, \min\{\mu_1, \mu_2\}) \cup (\max\{\mu_1, \mu_2\}, \infty)$$

there is a solution curve $\mathcal{T}_w$ of synchronized solutions (u, v) explicitly given by

$$U_\beta = \sqrt{\frac{\beta - \mu_2}{\beta^2 - \mu_1 \mu_2}}\, w \quad \text{and} \quad V_\beta = \sqrt{\frac{\beta - \mu_1}{\beta^2 - \mu_1 \mu_2}}\, w$$

where w is a positive solution of $\Delta w - w + w^3 = 0$.

- **Synchronization-implied uniqueness of positive solutions**

Theorem 1

(Wei-Yao, 2012) Assume Ω *is a radially symmetric domain in* $\mathbb{R}^n$ *with* $n = 1, 2, 3$*. Let* $N = 2$ *and* $\lambda_1 = \lambda_2$*. Then for* $\beta > \max\{\mu_1, \mu_2\}$*, any positive solution* (u, v) *must satisfy* u/v *is a constant, and consequently system* (1) *has a unique positive radial solution.*
In case $\mathbb{R}^1$*, and* $0 < \beta < \min\{\mu_1, \mu_2\}$*, system* (1) *has a unique positive solution.*

Rmk. 1) Synchronization implies the uniqueness of positive solution.
2) In general this is not the case for sign-changing solutions.

Synchronized partial symmetry of least energy solutions

Consider the least energy of one constraint for the N-system

$$c = \inf_{\mathcal{N}} I(\vec{u}).$$

$$\mathcal{N} = \{\vec{u} \neq (0, \dots, 0) \mid I'(\vec{u})\vec{u} = 0\}$$

Assume Ω is a radial domain.

Rmk. This may be achieved by semi-trivial solutions, one needs additional conditions to assure the existence of non-trivial vector solutions, e.g., Z. Liu-Wang (2010).

(Wang-Willem, 2014)

Theorem 2

Let $\vec{u} = (u_1, \ldots, u_N)$ be a minimizer of I on $\mathcal{N}$ with $u_j \neq 0$, $j = 1, \ldots, N$. Then there exists $e \in S^{n-1}$ such that u_j is **foliated Schwarz symmetric** *with respect to e for $j = 1, \ldots, N$.*

Here a function f defined on a radially symmetric domain Ω is said to be foliated Schwarz symmetric with respect to $e \in S^{n-1}$ if $f(x)$ only depends on $(r, \theta) = (|x|, \arccos(\frac{x}{|x|} \cdot e))$ and is non-increasing in θ.

Rmk. In case Ω is an annular domain or an exterior domain of a ball, the least energy solutions can be non-radial, and the components of the solutions have **'synchronized' foliated Schwarz symmetry** in the sense that all components of the solutions have the F.S.S. with respect to the same point.

The repulsive case.

Theorem 3

(Conti-Terracini-Verzini, 2002) Assume Ω *is a bounded domain in* $\mathbb{R}^n$ *with* $n = 1, 2, 3$*. Let* $N = 2$*,* $\lambda_1 = \lambda_2 = 0$*,* $\mu_1 = \mu_2 = 1$*. Then* $\beta \to -\infty$*, the least energy solution* (u, v) *converges to* (u_0, v_0) *satisfying that* $u_0 v_0 = 0$ *and* $u_0 - v_0$ *is a solution of* $-\Delta w = w^3$ *in* Ω *that changes exactly once.*

Rmk. 1. Segregation occurs here
2. De-focusing case, Chang, C.-S. Lin, T. Lin, W. Lin, 2004, segregated nodal domains with large repulsive couplings
3. Finer asymptotics for large repulsive couplings: Noris-Tavares-Terracini-Verzini (2010)
4. More on asymptotics, Dancer-K. Wang-Z. Zhang (2011-12);

Wei-Weth (2008), Dancer-Wei-Weth (2010)
studied the case $\lambda_1 = \lambda_2 = 1, \mu_1 = \mu_2 = 1$,
showed that for $\beta \leq -1$, the 2-system has **an unbounded sequence of positive solutions**.

In radial case, for each integer *k*, there exists a solution (u, v) such that $u - v$ has exactly *k* zeroes

The method is minimax by using the permutation symmetry of exchanging *u* and *v*

Rmk. again these are segregated solutions

T. Bartsch-N. Dancer-Wang (2010),
local and global bifurcations of positive solutions for the 2-system:
$(\lambda_1 = \lambda_2 = 1, \mathbf{0} < \mu_{\mathbf{1}} \leq \mu_{\mathbf{2}})$

$$\begin{cases} -\Delta u + u = \mu_1 u^3 + \beta v^2 u, \text{in } \Omega \\ -\Delta v + v = \mu_2 v^3 + \beta u^2 v, \text{in } \Omega \\ u = v = 0, \text{ on } \partial\Omega \end{cases} \tag{3}$$

We have a branch of synchronized positive solutions of (3)

$$\mathcal{T}_w := \{(\beta, U_\beta, V_\beta) : \beta \in (-\sqrt{\mu_1 \mu_2}, \mu_1) \cup (\mu_2, \infty)\}$$

Two other branches containing semi-trivial solutions:
$\mathcal{T}_1 = \{(\beta, \frac{w}{\sqrt{\mu_1}}, 0) : \beta \in \mathbb{R}\}$
$\mathcal{T}_2 = \{(\beta, 0, \frac{w}{\sqrt{\mu_2}}) : \beta \in \mathbb{R}\}$

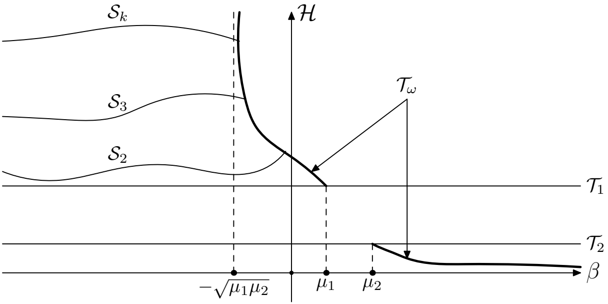
$$\mathcal{S} := \{(\beta, u, v) \in \mathbb{R} \times E \times E \setminus \mathcal{T}_w : (\beta, u, v) \text{ solves (3)}\}$$

We treat $\mathcal{T}_w$ as a trivial branch and consider local and global bifurcations emanating from $\mathcal{T}_w$.

Theorem 4

Suppose Ω is radial and bounded. Then for integer $k \geq 2$ there exist $\beta_k \in (-\sqrt{\mu_1\mu_2}, 0)$ such that $\beta_k \to -\sqrt{\mu_1\mu_2}$ and a connected set $\mathcal{S}_k \subset \mathcal{S}$ of solutions (β, u, v) of (3) such that

- $\overline{\mathcal{S}_k} \cap \mathcal{T}_w = \{(\beta_k, U_{\beta_k}, V_{\beta_k})\}$.
- *The projection $proj_1 : \mathcal{S}_k \to \mathbb{R}$ onto the parameter space satisfies $proj_1(\mathcal{S}_k) \supset (-\infty, \beta_k)$.*
- *For any $(\beta, u, v) \in \mathcal{S}_k$, the difference $(\mu_1 - \beta)^{1/2}u - (\mu_2 - \beta)^{1/2}v$ has precisely $k - 1$ zeroes.*



Nonradial positive solutions: synchronization and segregation

Consider the following system in $\mathbb{R}^3$

$$\begin{cases} -\Delta u + P(|x|)u = \mu_1 u^3 + \beta v^2 u, & x \in \mathbb{R}^3, \\ -\Delta v + Q(|x|)v = \mu_2 v^3 + \beta u^2 v, \end{cases} \tag{4}$$

where $P(r)$ and $Q(r)$ are positive radial potentials, $\mu_1 > 0, \mu_2 > 0$ and $\beta \in \mathbb{R}$ a coupling constant.

We assume that $P(r) > 0, Q(r) > 0$ satisfy the following conditions:

(P): There are constants $a \in \mathbb{R}$, $m > 1$, and $\theta > 0$, such that as $r \to +\infty$

$$P(r) = 1 + \frac{a}{r^m} + O(\frac{1}{r^{m+\theta}}). \tag{5}$$

(Q): There are constants $b \in \mathbb{R}$, $n > 1$, and $\epsilon > 0$, such that as $r \to +\infty$

$$Q(r) = 1 + \frac{b}{r^n} + O(\frac{1}{r^{n+\epsilon}}). \tag{6}$$

(S. Peng-Wang 2013)

Theorem 5

Let $n = 2, 3$. *There exists a sequence* $\{\beta_k\} \subset (-\sqrt{\mu_1\mu_2}, 0)$ *with* $\beta_k \to -\sqrt{\mu_1\mu_2}$ *as* $k \to \infty$ *such that for fixed* $\beta \in (-\sqrt{\mu_1\mu_2}, 0) \cup (0, \min\{\mu_1, \mu_2\}) \cup (\max\{\mu_1, \mu_2\}, \infty)$ *and* $\beta \neq \beta_k$ *for any* k, *problem (4) has infinitely many non-radial positive solutions* (u_ℓ, v_ℓ), *provided one of the following two holds:*

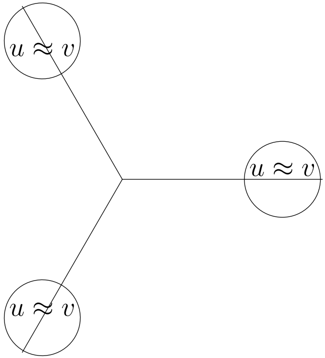
(i) $m < n$, $a > 0$ *and* $b \in \mathbb{R}$; *or* $m > n$, $a \in \mathbb{R}$ *and* $b > 0$;

(ii) $m = n$, $aB + bC > 0$, *where* $B > 0$, $C > 0$ *known constants.*

Furthermore, $\lim_{\ell\to\infty} \max u_\ell > 0$, $\lim_{\ell\to\infty} \max v_\ell > 0$, *and as* $\ell \to \infty$

$$||\sqrt{|\mu_1 - \beta|}u_\ell - \sqrt{|\mu_2 - \beta|}v_\ell||_{H^1} + |\sqrt{|\mu_1 - \beta|}u_\ell - \sqrt{|\mu_2 - \beta|}v_\ell|_{L^\infty} \to 0.$$

These are multi-bump type solutions, each component has a large number of bumps and the locations of the bumps for the two components are in ‘synchronization”
$\ell = 3$



(S. Peng-Wang 2013)

Theorem 6

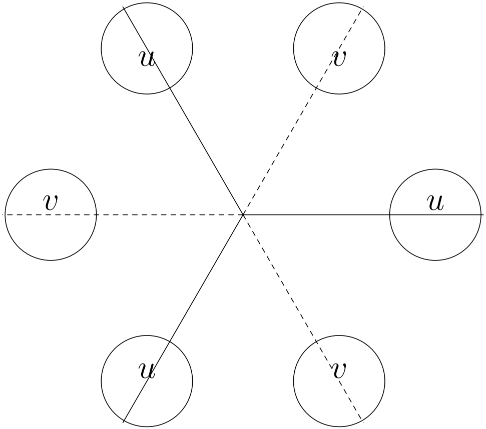
Let $n = 3$. *Suppose* $m = n$, $a > 0$, $b > 0$. *Then there exists* $\bar{\beta}^* > 0$ *such that, for* **fixed** $\beta < \bar{\beta}^*$, *problem (4) has infinitely many non-radial positive solutions* (u_ℓ, v_ℓ). *Furthermore,* $\lim_{\ell\to\infty} \max u_\ell > 0$, $\lim_{\ell\to\infty} \max v_\ell > 0$, *and as* $\ell \to \infty$

$$||\sqrt{\mu_2}u_\ell(\cdot) - \sqrt{\mu_1}v_\ell(T_\ell\cdot)||_{H^1} + |\sqrt{\mu_2}u_\ell(\cdot) - \sqrt{\mu_1}v_\ell(T_\ell\cdot)|_{L^\infty} \to 0.$$

Here $T_\ell \in SO(3)$ *is the rotation on the* (x_1, x_2) *plane of* $\frac{\pi}{\ell}$.

These are multi-bump type solutions, each component has a large number of bumps and the locations of the bumps for the two components are in ‘segregations’

$\ell = 3$



Following the idea of Wei-Yan (2010) for scalar equations
Here we show for systems the two components both have a large number of bumps at infinity, for the two components either the bumps coincide or separated by a shifting, depending on the coupling constants being attractive or repulsive.

Rmk. one of the potentials can be increasing, in scalar case there is no non-radial solutions

C.-S. Lin- S. Peng, preprint: Multi-bump type synchronized vector solutions for linearly coupled systems

Mixed couplings (interactions)

$$
\begin{aligned}
-\Delta u_1 + \lambda_1 u_1 &= \mu_1 u_1^3 + \beta_{\mathbf{12}} u_2^2 u_1 + \beta_{\mathbf{13}} u_3^2 u_1 \\
-\Delta u_2 + \lambda_2 u_2 &= \mu_2 u_2^3 + \beta_{21} u_1^2 u_2 + \beta_{\mathbf{23}} u_3^2 u_2 \\
-\Delta u_3 + \lambda_3 u_3 &= \mu_3 u_1^3 + \beta_{32} u_2^2 u_3 + \beta_{31} u_1^2 u_3
\end{aligned}
$$

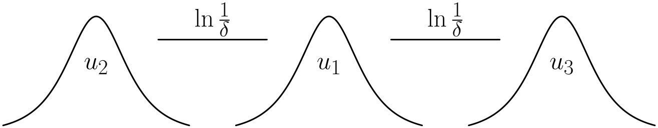
Question. partial synchronization and segregation?

Lin-Wei (2005), they considered a case of 3-systems:
$\lambda_1 = \lambda_2 = \lambda_3 = \mu_1 = \mu_2 = \mu_3 = 1$,
$\beta_{12} = \delta\hat{\beta}_{12} > 0, \beta_{13} = \delta\hat{\beta}_{13} > 0, \beta_{23} = \sqrt{\delta}\hat{\beta}_{23} < 0$.

Then for $\delta > 0$ small, there exists a bound state solution $u^\delta = (u_1^\delta, u_2^\delta, u_3^\delta)$ such that

$$u_1^\delta(y) \sim w(y), u_2^\delta(y) \sim w(y - R^\delta e_1), u_3^\delta(y) \sim w(y + R^\delta e_1)$$

where $e_1 = (0, 0, 1)$ and $R^\delta \sim \ln \frac{1}{\delta}$, w is the unique positive radial solution of $\Delta w - w + w^3 = 0$ in $\mathbb{R}^3$.


u_2
$\ln \frac{1}{\delta}$
u_1
$\ln \frac{1}{\delta}$
u_3

A preprint by N. Soave (2013):
simultaneous cooperation and competition for general
N-systems, by stretching out the repulsive constant to show the existence of vector positive solutions.

Joint work Y. Sato - Wang, preprints 2013

We consider a case of 3-systems on a bounded domain:
$\lambda_i > 0, \mu_i > 0$, $i = 1, 2, 3$.
$\beta := \beta_{12} > 0, \beta_{13} < 0, \beta_{23} < 0$.

We show for β large, there is a least energy positive solution, roughly speaking, u_1, u_2 are synchronized and kind of segregating from u_3.

Rmk. Co-existence of segregation and partial synchronizations

Theorem 7

There exists a $\beta_* > 0$ such that, for any $\beta > \beta_*$, there exists a least energy positive solution $\vec{u}_\beta = (u_{1,\beta}, u_{2,\beta}, u_{3,\beta})$. Moreover, as $\beta \to \infty$, there exist $U_j \in H_0^1(\Omega)$ ($j = 1, 2, 3$) such that

$$(\sqrt{\beta}u_{1,\beta}, \sqrt{\beta}u_{2,\beta}, u_{3,\beta}) \to (U_1, U_2, U_3) \quad \text{in } [H_0^1(\Omega)]^3.$$

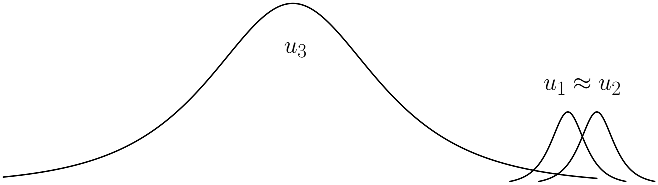
Here U_3 is a positive least energy solution of

$$-\Delta u_3 + \lambda_3 u_3 = \mu_3 u_3^3 \quad \text{in } \Omega,$$

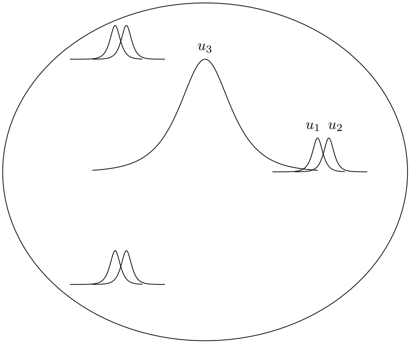
and (U_1, U_2) is a positive least energy solution of

$$-\Delta u_1 + (\lambda_1 - \beta_{1,3} U_3^2) u_1 = u_1 u_2^2,$$

$$-\Delta u_2 + (\lambda_2 - \beta_{2,3} U_3^2) u_2 = u_1^2 u_2, \quad \text{in } \Omega.$$


u_3
$u_1 \approx u_2$

When Ω is a ball, we may construct vector solutions with the first two components have partially synchronized multi-peaks while segregated from the third component.
But the number of peaks is limited from above.



Idea for the proof of the least energy solutions.
Recall $\beta := \beta_{12}$ and define

$$c_\beta = \inf_{u \in M_\beta} \sum_{i=1}^{3} ||u_i||^2_{\lambda_i}$$

$$M_\beta = \{u \ \ | \ f_1(u) + f_2(u) = 0, f_3(u) = 0, (u_1, u_2) \neq (0,0), u_3 \neq 0\}$$

$$f_i(u) = \nabla_{u_i} I(u) u_i = ||u_i||^2_{\lambda_i} - \mu_i |u_i|^4_4 - \sum_{j \neq i} \beta_{ij} |u_i u_j|^2_2.$$

for β small, this minimization produces only semi-trivial solutions,

for β not so large, there is no positive solutions for β large we show the minimizer is positive with each component by showing the least energy is less than the energies of semi-trivial solutions

Let $d_i = \inf_{||u_i||^2_{\lambda_i} \le \mu_i |u|^4_4} ||u||^2_{\lambda_i}$. Then $4d_i$ is the least energy of the least energy solution of $-\Delta u + \lambda_i u = \mu_i u^3$.

The key estimates are

a) as $\beta \to \infty$ we have

$$c_\beta \le d_3 + \frac{e}{\beta} + o(\frac{1}{\beta}).$$

where e is the least energy of the limiting system for the first two components.

b) if $I(u) < d_3 + \min\{d_1, d_2\}$, then $u_i \neq 0$ for $i = 1, 2, 3$.

c) asymptotic profile of solutions for large β

In Summary

• for attractive case, the solutions tend to be of synchronized type, radial or non-radial

• For repulsive case, the solutions tend to be segregated type, and the question is multiple positive solutions, rich bifurcation structures.

• With mixed couplings, the situation becomes more complicated and there may be co-existence of synchronizations and segregations.

Acknowledgment of co-authors:

Thomas Bartsch (Giessen),

Norman Dancer (Sydney),

Shuangjie Peng (Central China Normal Univ),

Yohei Sato (Osaka City Univ),

Michel Willem (Louvain La Neuve).