Rate of Convergence to Separable Solutions of the Fast Diffusion Equation

Marek Fila

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Consider

$$\begin{cases} u_\tau = \nabla \cdot (u^{m-1}\nabla u), & x \in \mathbb{R}^n,\ \tau \in (0, T), \\ u(x, 0) = u_0(x) \geq 0, & x \in \mathbb{R}^n, \end{cases}$$

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conservation of mass $\quad \Leftrightarrow \quad m \geq \frac{n-2}{n}$

$$\int_{\mathbb{R}^n} u_0(x)\,dx = C < \infty \Rightarrow \int_{\mathbb{R}^n} u(x,\tau)\,dx = C$$

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$$u(x,\tau) := ((1-m)(T-\tau))^{\frac{1}{1-m}} \varphi^{\frac{1}{m}}(x)$$

is a separable solution of $u_\tau = \Delta(u^m)$ if φ satisfies

$$\Delta\varphi + \varphi^p = 0, \qquad x \in \mathbb{R}^n, \qquad p := \frac{1}{m}.$$

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Do other solutions with extinction behave like separable solutions?

change of variables:

$$v(x,t) := ((1-m)(T-\tau))^{-\frac{m}{1-m}} u^m(x,\tau), \qquad t := -\frac{1}{1-m} \ln \frac{T-\tau}{T}\,.$$

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If u is a solution with extinction at $\tau = T$ then v satisfies

$$\begin{cases} (v^p)_t = \Delta v + v^p, \qquad x \in \mathbb{R}^n,\ t > 0, \qquad p = \frac{1}{m}\,, \\ v(x,0) = v_0(x) := ((1-m)T)^{-\frac{m}{1-m}} u_0^m(x), \qquad x \in \mathbb{R}^n. \end{cases}$$

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Vázquez 2006

Let $n > 2$, $m = (n-2)/(n+2)$, $u_0 \in L^{2n/(n+2)}(\mathbb{R}^n)$.

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Let $n > 2$, $m = (n-2)/(n+2)$, $u_0 \in L^{2n/(n+2)}(\mathbb{R}^n)$.
Then there exist $\lambda > 0$ and $x_0 \in \mathbb{R}^n$ such that

$$(T-\tau)^{-(n+2)/4} u(x,\tau) =$$

$$(4n)^{(n+2)/4} \left(\frac{\lambda}{\lambda^2 + |x - x_0|^2} \right)^{(n+2)/2} + \vartheta(x,\tau),$$

$$\sup_{x \in \mathbb{R}^n} (1 + |x|^{n+2}) |\vartheta(x,\tau)| \to 0 \qquad \text{as} \quad \tau \to T.$$

For a smoothly bounded domain $\Omega \subset \mathbb{R}^n$ all bounded positive solutions of

$$u_\tau = \Delta(u^m)$$

with the homogeneous Dirichlet boundary condition and

$$(n-2)_+/(n+2) < m < 1$$

extinguish in finite time and they approach separable solutions (Berryman & Holland 1980, Kwong 1988, Savaré & Vespri 1994, Bonforte, Grillo & Vázquez 2012, Akagi & Kajikiya 2013).

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Little is known about the convergence rate, only upper bounds for the decay rates of the entropy and of a weighted L^2-norm of the relative distance from the separable solution are given in [BGV] for $m = 1 - \varepsilon$.

There is a family of positive radial solutions of

$$\Delta\varphi + \varphi^p = 0, \qquad x \in \mathbb{R}^n,$$

if and only if $p \geq p_S$,

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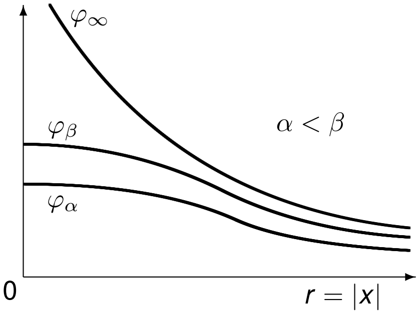
We denote a radial steady state by $\varphi = \varphi_\alpha(r)$, $r = |x|$, $\alpha > 0$, where $\varphi_\alpha(r)$ satisfies

$$\begin{cases} (\varphi_\alpha)_{rr} + \dfrac{n-1}{r}(\varphi_\alpha)_r + (\varphi_\alpha)^p = 0, & r > 0, \\ \varphi_\alpha(0) = \alpha, \quad (\varphi_\alpha)_r(0) = 0. \end{cases}$$

For each $\alpha > 0$, the solution φ_α is decreasing in r and $\varphi_\alpha(r) \to 0$ as $r \to \infty$.



$$p \geq p_c > p_S,$$

$$p_c := \begin{cases} \dfrac{(n-2)^2 - 4n + 8\sqrt{n-1}}{(n-2)(n-10)} & \text{if } n > 10, \\ \infty & \text{if } n \leq 10. \end{cases}$$

$$\lim_{\alpha \to 0} \varphi_\alpha(|x|) = 0, \qquad \lim_{\alpha \to \infty} \varphi_\alpha(|x|) = \varphi_\infty(|x|),$$

where φ_∞ is a singular steady state given by

$$\varphi_\infty(|x|) := L|x|^{-\nu}, \quad x \in \mathbb{R}^n \setminus \{0\},$$

with

$$\nu := \frac{2}{p-1}\,, \qquad L := \Big\{\nu\,(n-2-\nu)\Big\}^{1/(p-1)}.$$

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$$\varphi_\alpha(|x|) = L|x|^{-\nu} - a_\alpha |x|^{-\nu-\lambda_1} + o\Big(|x|^{-\nu-\lambda_1}\Big) \qquad \text{as } |x| \to \infty,$$

$$\lambda_1 = \lambda_1(n,p) := \frac{n-2-2\nu - \sqrt{(n-2-2\nu)^2 - 8(n-2-\nu)}}{2}\,,$$

$a_\alpha > 0$.

(Gui, Ni & Wang 1992).

Thm 1. (F. & WINKLER)

Let $n > 10$, $p > p_c$, $\alpha > 0$, and assume that v_0 is continuous in $\mathbb{R}^n$ and satisfies

$$0 \le v_0(x) \le L|x|^{-\nu}, \qquad x \in \mathbb{R}^n \setminus \{0\},$$

$$|v_0(x) - \varphi_\alpha(|x|)| \le b|x|^{-\gamma}, \qquad x \in \mathbb{R}^n, \quad |x| > 1,$$

with some $b > 0$ and

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$$\boxed{|v(x,t) - \varphi_\alpha(|x|)| \le Ce^{-\kappa(\gamma)t}, \qquad x \in \mathbb{R}^n, \quad t \ge 0,}$$

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$$\nu + \lambda_1 \to (n-2)/2 \qquad \text{as} \qquad p \to p_c\,.$$

Corollary

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Then for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\boxed{|v(x,t) - \varphi_\alpha(|x|)| \le C_\varepsilon e^{-(\kappa^* - \varepsilon)t}, \qquad x \in \mathbb{R}^n, \quad t \ge 0,}$$

where

$$\kappa^* := \kappa((n-2)/2) = \frac{(n-2)^2}{4pL^{p-1}} - 1 > 0\,.$$

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$$\boxed{|v(0,t) - \varphi_\alpha(0)| \geq Ce^{-\kappa(\gamma)t}, \qquad t \geq 0.}$$

Thm 3. (F. & WINKLER)

Let $n > 2$, $p_S \le p < p_c$, $\alpha > 0$, and assume that v_0 is bounded and continuous in $\mathbb{R}^n$, $v_0 \not\equiv \varphi_\alpha$.

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(i) If

$$v_0(x) \geq \varphi_\alpha(|x|), \qquad x \in \mathbb{R}^n,$$

then

$$\|v(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \to \infty \qquad \text{as} \quad t \to \infty.$$

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(ii) If

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then there is $T \in (0,\infty]$ such that

$$\|v(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \to 0 \qquad \text{as} \quad t \to T.$$

If $p = p_S$ then a solution converging to φ_α intersects φ_α for all $t > 0$.

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κ is determined by the decay rate of $v_0 - \varphi_\alpha$ (by γ).

Compare with the Fujita equation

$$\begin{cases} v_t = \Delta v + v^p, & x \in \mathbb{R}^n,\ t > 0, \qquad p > p_c\,, \\ v(x,0) = v_0(x), & x \in \mathbb{R}^n, \end{cases}$$

F., Winkler & Yanagida 2005, Hoshino & Yanagida 2008:

$$v_0 - \varphi_\alpha \sim r^{-\gamma}, \gamma \in (\nu + \lambda_1, \nu + \lambda_2 + 2) \quad \Rightarrow \quad \|v(\cdot,t) - \varphi_\alpha\|_\infty \sim t^{-\frac{\gamma-\nu-\lambda_1}{2}}.$$

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$$\begin{cases} (v^p)_t = \Delta v + v^p, & x \in \mathbb{R}^n,\ t > 0, \qquad p > p_c\,, \\ v(x,0) = v_0(x), & x \in \mathbb{R}^n, \end{cases}$$

$$v_0 - \varphi_\alpha \sim r^{-\gamma}, \gamma \in (\nu+\lambda_1, (n-2)/2) \quad \Rightarrow \quad \|v(\cdot,t) - \varphi_\alpha\|_\infty \sim e^{-\kappa(\gamma)t},$$

$\kappa(\gamma)$ is an explicit quadratic function of γ.

a different change of variables

Let u be a solution of $u_\tau = \nabla \cdot (u^{m-1} \nabla u)$ with extinction at $\tau = T$.

$$R(\tau) := (T-\tau)^{-\beta}, \qquad \beta := \frac{\mu}{2(n-\mu)}, \qquad \mu := \frac{2}{1-m},$$

$$t := \frac{1}{\mu} \log\left(\frac{R(\tau)}{R(0)}\right), \qquad y := \sqrt{\frac{\beta}{\mu}} \frac{x}{R(\tau)}, \qquad w(y,t) := R^n(\tau) u(x,\tau),$$

$$w_t = \nabla \cdot (w^{m-1} \nabla w) + \mu \nabla \cdot (yw), \qquad t > 0, \quad y \in \mathbb{R}^n.$$

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steady states

$$W_D(y) := (D + |y|^2)^{\frac{1}{m-1}}, \qquad D \geq 0.$$

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If $m = 1/p_S$ and u is separable then, as $t \to \infty$, $w(0,t) \to \infty$, $w(y,t) \to 0$ for $y \neq 0$.

Thank you for your attention.