Global/Local Conjectures in Representation Theory of Finite Groups

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1 Overview of the Field

Group Theory is essentially the theory of symmetry for mathematical and physical systems, with major impact in diverse areas of mathematics. The Representation Theory of Finite Groups is a central area of Group Theory, with many fascinating and deep open problems, and significant recent successes. In 1963 R. Brauer [B] formulated a list of deep conjectures about ordinary and modular representations of finite groups. These conjectures have led to many new concepts and methods, but basically all of his main conjectures are still unsolved to the present day. A new wealth of difficult problems, relating global and local properties of finite groups, was opened up in the seventies and eighties (of the 20th century) by subsequent conjectures of J. McKay [Mc], J. Alperin [A2], M. Broué [Br], and others, all remain open up to date.

The classification of finite simple groups raised the hope that one should be able to reduce some of the aforementioned conjectures to statements about simple groups, and subsequenly establish these statements by exploring deep knowledge about simple groups provided by the Deligne-Lusztig theory and other recent fundamental results in group theory. This hope has recently materialized when the first three of the above mentioned conjectures have been reduced to questions about finite simple groups (see M. Isaacs, Malle, and G. Navarro [IMN], Navarro and P.H. Tiep [NT2], Navarro and Späth [NSp], and B. Späth [Sp2, Sp3]). Since then, further substantial progress on some of the remaining steps towards proving these conjectures has been achieved.

2 Recent Developments and Open Problems

In the meeting, we concentrated on recent progress on the following famous and longstanding global/local conjectures, some of which is discussed in more detail below. If G is a finite group, we denote by $\mathrm{Irr}(G)$ the set of irreducible complex characters of G and by $\mathrm{Irr}_{p'}(G)$ its subset consisting of characters having degree not divisible by p.

2.1 The Alperin-McKay Conjecture and refinements

The McKay conjecture [Mc] from 1972 is at the center of the representation and character theory. It is the origin, together with the Alperin Weight Conjecture, of the more far-reaching Dade conjecture as well as the

Broué Conjecture (see below). If p is a prime and G is a finite group, then the McKay conjecture asserts that

$$|\mathrm{Irr}_{p'}(G)| = |\mathrm{Irr}_{p'}(\mathbf{N}_G(P))|\,,$$

where P is a Sylow p-subgroup of G and $\mathbf{N}_G(P)$ its normalizer. That is to say, a certain fundamental information on G is encoded in some local subgroup of G, namely the Sylow normalizer. Some time later, Alperin [A1] extended the statement to include Brauer blocks. This generalization is now known as the Alperin-McKay conjecture. Later, Isaacs, Navarro, and A. Turull [IN2, N, Tu] discovered several refinements of the conjecture and these have contributed significantly to a further understanding of the problem.

A significant breakthrough was achieved by Isaacs, Malle and Navarro in 2007 in [IMN] where they reduced the McKay conjecture to a question on simple groups, the so-called *inductive McKay condition*. The latter has been established for important families of simple groups by Malle and Späth [Ma1, Sp1], relying heavily on the Deligne-Lusztig theory [L], and has led to interesting and difficult questions on automorphism groups of simple groups of Lie type. Ongoing work of Broué, P. Fong, and B. Srinivasan aims at proving the McKay conjecture and its refinements for finite reductive groups (in cross characteristic). Very recently, M. Cabanes and Späth [CS1, CS2] have proved the inductive McKay condition for some further families of simple groups of Lie type, and some of these results were presented by Cabanes at the meeting, see Section 3 for details. [1]

2.2 The Alperin Weight Conjecture

If the McKay Conjecture counts characters of p'-degree, then the Alperin Weight Conjecture (AWC), as formulated by R. Knörr and one of our main intended speakers G. R. Robinson [KR], counts characters with maximal p-part, the so called defect zero characters. Specifically, a character $\chi \in \mathrm{Irr}(G)$ has defect zero if $\chi(1)_p = |G|_p$. Alperin's original 1986 formulation [A2] is given by the formula

$$l(G) = \sum_Q z(\mathbf{N}_G(Q)/Q)\,,$$

where Q runs over representatives of conjugacy classes of p-subgroups in G; furthermore, $z(X)$ is the number of defect zero characters of the group X, and $l(G)$ is the number of p-regular classes of G. By using Möbius inversion, it is then possible to describe $z(G)$ in terms of local subgroup information. E. Dade's Conjecture extends the Knörr–Robinson formulation of AWC, and suggests a way to count the characters of G of any defect in terms of the local subgroup structure. It therefore generalizes both the McKay Conjecture and the Alperin Weight Conjecture. Dade's Conjecture was proved for p-solvable groups by Robinson [R].

In 2011 Navarro and Tiep [NT2] reduced the Alperin Weight Conjecture to a question on simple groups, the so-called *inductive AWC condition*. (L. Puig also announced another reduction of AWC to simple groups). This inductive AWC condition has now been established by work of Navarro–Tiep, Malle, J. An and H. Dietrich for many families of finite simple groups. Following the work of Navarro and Tiep [NT2], Späth [Sp2] has proved a reduction theorem for the stronger blockwise version of the Alperin Weight Conjecture. In the basic case of cyclic defect blocks, these inductive conditions (also for the Alperin-McKay conjecture) have been established recently by Koshitani and Späth [KS] and presented at the meeting.

2.3 The Brauer Height Zero Conjecture

The Height Zero Conjecture, a fundamental problem on blocks of finite groups formulated in 1955 by Brauer, lead to many interesting and challenging questions on characters of finite groups. If B is a p-block with defect group D, then Brauer conjectured that all irreducible characters in B have height zero if and only if D is abelian. In particular, a positive solution would exhibit another connection between local and global invariants, providing, for example, an extremely simple method to detect from a group's character table whether its Sylow p-subgroup is abelian. The p-solvable case of the Height Zero Conjecture is already an impressive result by D. Gluck and T. Wolf [GW].

In the last two years, several breakthrough results have been proved concerning this conjecture. First, the solution of the Brauer Height Zero Conjecture for 2-blocks of maximal defect by two of the proposers,

[1] Dr. B. Späth was one of the main invited speakers. Due to unexpected circumstances, she could not attend the meeting.

Navarro and Tiep, in [NT1]. Next, R. Kessar [2] and another proposer, Malle, completed in [KM] the proof of the "if" direction of the conjecture, relying in particular on the Berger-Knörr reduction [BK], as well as established the "only if" implication for the quasisimple groups. In another direction, Navarro and Tiep [NT3] have been able to prove the odd-p analogue of the aforementioned theorem of Gluck and Wolf, and therewith remove one of the main obstacles towards establishing the remaining, "only if" direction of the Height Zero Conjecture. Navarro and Späth [NSp] in 2012 succeeded in proving a reduction theorem for this direction of the conjecture to the inductive Alperin–McKay condition. Thus progress on the aforementioned Alperin–McKay Conjecture will ultimately also lead to a full proof of Brauer's conjecture.

2.4 The Broué Conjecture

The Alperin-McKay conjecture asserts that if B is a Brauer block of a finite group, then B and its First Main Theorem correspondent b have the same number of height zero characters. The block b is a uniquely determined block of the local subgroup $\mathbf{N}_G(D)$, where D is the defect group of B. Now, blocks are algebras, and Broué [Br] conjectured that the algebras B and b have intimate structural connections that should imply the desired facts on height zero characters and much more. Currently, the Broué Conjecture is only stated for D abelian, and it remains a challenge to find the correct formulation when the defect group is non-abelian.

In the landmark paper [CR1] J. Chuang and Rouquier have given a proof of the Broué Abelian Defect Conjecture for symmetric groups. More recently, D. Craven and Rouquier [CR2] have been exploring an interesting connection between cohomology of Deligne-Lusztig varieties and the Broué Conjecture for unipotent blocks of finite groups of Lie type. This approach already proves Broué's conjecture in some new cases and helps determine the previously unknown structure of the Brauer tree for some blocks with cyclic defect. Furthermore, at the meeting Craven announced a reduction theorem for the case of principal blocks of the Broué conjecture.

2.5 Future Directions

Recent progress on all these fundamental conjectures raises the hope that complete proofs of some of them may be possible in the not too distant future. Further significant progress will be achieved once we can resolve a number of basic questions on representations of finite groups of Lie type. With many experts in the Deligne-Lusztig theory present at the meeting, we had a discussion session on Thursday afternoon to draw a roadmap towards possible solutions of some of these principal obstacles.

3 Presentation Highlights

Now that reduction theorems for many basic conjectures have been proved, major efforts are concentrated in establishing the resulting inductive conditions for various families of finite simple groups. The opening lecture was given by M. Cabanes, in which he reported about his recent joint work with B. Späth establishing the inductive McKay conditions for finite simple groups of Lie type A (projective special linear and unitary groups over finite fields). Building on this, they also showed how one can verify the more involved conditions on the inductive Alperin-McKay conditions.

In a related direction, recent work of S. Koshitani and Späth is devoted to proving the inductive Alperin-McKay and blockwise Alperin weight conditions for cyclic defect blocks. Koshitani spoke at the meeting about this result.

When the block in question has a relatively small defect group, it is possible to prove some of the basic conjectures for this block directly, without using reduction theorems and the classification of the finite simple groups. In his talk, B. Sambale presented a new result establishing the Alperin–McKay Conjecture for blocks with metacyclic, minimal non-abelian defect groups. These are precisely the metacyclic groups whose derived subgroup have order p. In the special case $p = 3$, he also verified the Alperin Weight Conjecture for these defect groups.

M. Linckelmann spoke about his joint work with R. Kessar and Navarro. The starting point is a result by M. Isaacs, showing that a finite group G is p-nilpotent if and only if the highest power of p dividing the sum

[2] Prof. R. Kessar was one of the main invited speakers. Unfortunately, due to visa issues, she was not able to make the trip.

of the squares of the characters of degree prime to p is equal to the index of the p-focal subgroup in a Sylow p-subgroup of G. They showed that this statement admits a blockwise version which gives a characterization of nilpotent blocks. They also stated a variation of Brauer's Height Zero Conjecture for hyperfocal subalgebras.

For special classes of finite groups, the putative bijection in the McKay conjecture can indeed be constructed in a canonical way. This is the case for finite groups G with the property that $N_G(P) = PC_G(P)$ for a Sylow p-subgroup P and $p > 2$. At the meeting, C. Vallejo discussed her joint work with G. Navarro and P. H. Tiep, in which they proved that there is a natural correspondence between the irreducible characters of G of degree not divisible by p in the principal block and those of $N_G(P)$. In particularly, their result yields a natural McKay bijection for groups with self-normalizing Sylow p-subgroups. In fact, for these groups, Navarro's refinement of the McKay conjecture implies that there is a way to read off from the character table of G whether a Sylow p-subgroup of G is self-normalizing. This is indeed the case when $p > 2$, by a result of Navarro, Tiep, and A. Turull [NTT]. The case $p = 2$ of this question remains open, and A. Schaeffer Fry's talk discussed the reduction of that problem to simple groups.

Another highlight was O. Dudas' lecture, discussing his joint work with Malle on new techniques to compute decomposition numbers for finite groups of Lie type in non-defining characteristic. These techniques are based on modular analogues of Lusztig's results on the cohomology of Deligne-Lusztig varieties, some of which are still conjectural. Potential applications of these techniques include (i) a definition of families for representation in positive characteristic, (ii) an approach to showing unitriangular shape of the decomposition matrix, and (iii) bounds on the number of (Morita classes of) unipotent blocks depending only on the rank of the underlying algebraic group.

More generally, further progress in both complex and cross characteristic representation theories of finite groups of Lie type depends on better understanding of character sheaves, a class of perverse sheaves on a connected reductive algebraic group G as introduced by Lusztig in 1985. Given a Steinberg endomorphism $F : G \to G$, each F-stable character sheaf admits a class function of the finite group G^F. Determining the values of these class functions is of central importance to the determination of the ordinary character table of the finite group G^F. In his talk, J. Taylor presented a recent result which gives the restriction of each such class function to the set of unipotent elements. He then described how this fits into a larger goal of understanding Kawanaka's generalised Gelfand-Graev representations and the decomposition numbers of G^F in non-defining characteristic.

Harish-Chandra series of Brauer characters in groups of Lie type have been introduced more than 20 years ago. By work of Geck, the unipotent Brauer characters of the finite unitary groups $\mathsf{GU}_n(q)$ are labelled by partitions of n. The question of describing the division of the unipotent Brauer characters of the unitary groups into Harish-Chandra series is still open in general. G. Hiss' talk presented a series of conjectures relating this division with crystal graphs of certain integrable highest weight modules of the quantum group corresponding to a suitable affine Lie algebra of type A, and gave some evidence for these conjectures.

Broué, Malle and Michel [BMM] gave a bijection between between characters in a block of a general linear or classical group on one hand, and characters of a corresponding complex reflection group on the other hand. In her talk, B. Srinivasan gave a description of that connection, as arising from two equivalent representations of a Heisenberg algebra acting on a Fock space. She also made some remarks on the ℓ-decomposition numbers of $\mathrm{GL}_n(q)$ for large ℓ, again using the Fock space.

Given the recent progress on the Brauer Height Zero Conjecture, characters of positive defect in p-blocks of finite groups have started to gain more attention. In particular, a conjecture of Eaton and Moreto states that, whenever the defect group of a block is non-abelian, the smallest non-zero height in the block and in its defect group are the same. O. Brunat spoke about his joint work with Malle, where they verified this conjecture for all principal 2-blocks of finite quasi-simple groups, for all blocks of groups of Lie type when the prime is the defining characteristic and for all covering groups of symmetric and alternating groups.

Many years ago, Brauer and Feit proved that the irreducible characters in a block of defect $d \geq 2$ have height at most $d - 2$. Their argument used Brauer's notion of "subsections". It turns out the same result can be obtained using vertices and sources, as shown in M. Geline's talk.

D. Craven presented his recent joint work with R. Rouquier on the Broué conjecture. With the classification of Brauer trees for unipotent blocks of groups of Lie type being completed, they aim to investigate what happens in the non-cyclic defect group case. In particular, Craven discussed the techniques needed to solve the Brauer tree case, which of these techniques can extend, and how the derived category of a unipotent

ℓ-block of $G(q)$ is independent not only of q, but also of ℓ.

Although being in the focus of intensive study for more than two decades, verifying the Broué conjecture for many of the 26 sporadic finite simple groups is still very challenging, both theoretically and computationally. J. Müller talked about his joint work with Koshitani and Noeske (Aachen), in which they showed how a combination of theoretical strategies and techniques from computational representation theory can be pursued to prove the Abelian Defect Group Conjecture of Broué for some of the sporadic simple groups.

Broué's perfect isometries are the shadow at the level of characters of derived equivalences between blocks of finite groups. In 1988, Michel Enguehard proved that two blocks of symmetric groups with the same weight are perfectly isometric. J.-B. Gramain discussed his recent joint work with O. Brunat, in which they generalize this result to many families of groups, such as alternating groups, double covers of symmetric and alternating groups, Weyl groups of types B and D, and certain wreath products.

An important step in the celebrated Chuang–Rouquier proof of the Broué conjecture for symmetric groups in characteristic p is a theorem of Chuang and Kessar establishing a Morita equivalence between the wreath product of the principal block of the symmetric group $\mathfrak{S}_p$ with $\mathfrak{S}_w$ and a certain special block of weight w of a symmetric group, called RoCK block (or Rouquier block), for $w < p$. In an effort to generalise this result to the case $w \geq p$ (i.e. to blocks of non-abelian defect), Turner conjectured that in this general case the same wreath product is Morita equivalent to a certain idempotent truncation of the RoCK block. A. Evseev's talk outlined a proof of Turner's conjecture, which also gives a new (and more explicit) proof of the Chuang–Kessar theorem.

Given any finite group G and any prime p, it is interesting to ask which ordinary irreducible representations of G remain irreducible in characteristic p. This question was solved for the symmetric groups several years ago, by the work of G. James and others. In his talk, M. Fayers addressed the case of the alternating group when p is odd. He explained how one can translate the question to one about representations of the symmetric group and thus ultimately answer it.

Motivated by the Broué conjecture and splendid Rickard equivalences, R. Boltje and P. Perepelitsky introduced the notion of p-permutation equivalences between p-blocks. In his talk at the meeting, Boltje discussed their joint work where they proved that a p-permutation equivalence between two blocks A and B preserves important invariants of the blocks A and B. Namely, such an equivalence induces (i) an isomorphism between defect groups and fusion systems, (ii) p-permutation equivalences on the local levels and an isotypy between A and B, (iii) Morita equivalences between the Brauer correspondents of A and B. It also preserves the 2-cohomology classes on the inertia groups and also the Külshammer–Puig classes for corresponding centric subgroups of the defect groups. They also determined restrictive properties of the shape of a p-permutation equivalence, and showed that the group of p-permutation auto-equivalences of a block is finite, and determined its structure in very simple cases.

Yet another highlight was B. Elias's lecture about his joint work with G. Williamson on Soergel bimodules and the p-canonical basis. In 1990, Soergel introduced a monoidal category SBim of bimodules over a polynomial ring, which encapsulates category O for the complex semisimple Lie algebra attached to a given semisimple algebraic group G. In 2000, he showed that this same monoidal category, but now defined over a field k of finite characteristic, encapsulates the rational representation theory of the algebraic group $G(k)$ "around the Steinberg weight." Both the characteristic zero and characteristic p versions have Grothendieck group isomorphic to the Hecke algebra H of the Weyl group, and the symbol of an object, when expressed in the standard basis of H, encodes certain multiplicities. In characteristic zero, Elias and Williamson have (re)proved the Soergel conjecture, which states that the symbols of indecomposable bimodules yield Kazhdan–Lusztig's canonical basis of H. In characteristic p, the symbols of indecomposable bimodules yield a basis called the p-canonical basis; its computation is an interesting and difficult open problem. In a more recent joint work, extending earlier work of Libedinsky and Elias–Khovanov, they established a presentation of (a monoidal subcategory of) SBim by generators and relations. This allows for a direct algebraic approach to the computation of the p-canonical basis, an approach which has already led to Williamson's recent disproof of the Lusztig conjecture.

A celebrated theorem of C. Jordan states that, for any d there exists a constant C such that any finite subgroup G of $GL_d(\mathbb{C})$ has an abelian subgroup $A \leq G$ of index at most C, and A can be generated by at most d elements. While the theorem is about finite linear groups, a number of analogues of it have been proposed. In his talk, A. Turull discussed his joint work with I. Mundet i Riera in which some of these

conjectures are reduced to simpler questions. These joint results are used by Mundet to prove a conjecture of É. Ghys on manifolds without odd cohomology.

4 Scientific Progress Made and Outcome of the Meeting

4.1 General Comments

The meeting featured 20 talks, given by well-known experts in the area as well as many younger participants (including postdocs and a Ph. D. student !) The main theme of the meeting was to discuss recent important progress on several old and famous conjectures in modular representation theory, like the McKay conjecture, the Alperin weight conjecture, and the Brauer Height Zero Conjecture.

Aside from the officially scheduled talks, ample time was allocated to informal discussions. As mentioned above, a discussion session took place on Thursday afternoon to draw a roadmap towards possible solutions of some of these principal obstacles on the way to complete proofs of these conjectures. Many participants commented that the workshop "was a very fruitful and inspiring meeting due to the presence of almost all experts of the field, plus some other experts in neighboring subjects." Some of the younger participants also said that "meeting all the renowned experts in the field was very inspirational" and "through this workshop they had fruitful discussions which will be determining for their future research". Let us also quote Paul Fong, another participant, who said "I thought the recent Banff workshop one of the best of its kind. The theme of the workshop was well-served by the depth and breadth of the talks, by the insightful questions that followed many talks, and by the discussion sections Thursday afternoon. I liked the mixture of old and new faces. That so many young mathematicians are pushing the frontiers of the field is really encouraging. The relaxed atmosphere of the workshop gave ample opportunities for participants to have exchanges on the known and unknown."

4.2 Collaboration Started or Continued During the Meeting

Numerous additional discussions between the meeting's participants led to further collaborations and new results and projects; we reproduce some of the comments on these outcomes given by participants below.

Boltje: "It was a wonderful experience to be at BIRS. The following developments would not have happened (at all or as quickly) without the visit to Banff: (1) After my talk, Jürgen Müller and Susanne Danz approached me with the suggestion to write a computer program that would allow to compute and analyze p-permutation equivalences and more generally p-permutation modules. We met for about 3 hours, discussed possible algorithms, and have now an initial concept to work on. (2) There is a logical hierarchy between different types of equivalences of blocks of finite groups. Discussions with Jeremy Rickard and Markus Linckelmann gave me new ideas how to construct p-permutation equivalences that do not come from the stronger notion of a splendid Rickard equivalence. Examples of this type are currently unknown. (3) In collaboration with Susanne Danz we continued to complete a paper that we started more than a year ago on the quasi-heredity of certain category algebras."

Cabanes: "With Tiep I discussed the possibility of proving certain stabilizer statements in the case of finite symplectic groups from the existing literature he knows well. With Brunat, following Gramain's talk on their joint work, I discussed the possible applications of the representation theory of wreath products and the related Morita equivalences to the proof of more equivalences between blocks of Weyl groups or Hecke algebras of classical types. I had many discussions with Paul Fong about the relevance of his joint work with Broué and Srinivasan on Dade's conjecture in the case of unipotent blocks of finite reductive groups, especially in view of recent results by Späth producing a reduction theorem for that conjecture."

Danz: "Inspired by Robert Boltje's talk on Tuesday, he, Jürgen Müller, and I started to discuss possible strategies for investigating p-permutation equivalences between blocks of finite groups computationally. During the problem session on Thursday afternoon we developed some concrete ideas, and we now plan to meet again this summer to pursue our joint project. Moreover, during the free afternoon on Wednesday, Boltje and I had the chance to make significant progress on another joint project (on quasi-hereditary twisted category algebras), which we had started several months ago, but were not able to complete just via email or skype correspondence."

Denoncin: "Through this workshop I had fruitful discussions in particular with Brunat, Chuang, Gramain and Olsson which will be determinant for my future research. In particular I believe that I have all the tools to prove that the extension of Enguehard's isometry I constructed between two blocks of double cover of symmetric groups in characteristic 2 is perfect. The next goal is to do the same with double covers of alternating groups, thus completing the recent work of Brunat–Gramain."

Eaton: "Here are some fruitful interactions I had during the conference: (i) Initiation of research with Benjamin Sambale on invariants of 2-blocks with elementary abelian defect groups. This is in its early stages, but we have made some progress. (ii) Discussion on upper bounds of Loewy lengths of blocks with Shigeo Koshitani. We were able to discuss at length my ideas for these bounds. (iii) Discussions with Markus Linckelmann on two topics: centres of blocks; and subrings of blocks defined by irreducible characters of height zero. The former is leading to a new line of enquiry for my PhD student, involving the projective centre of a block and stable equivalences; the latter is related to his talk at BIRS and relates to my past research on generalised perfect isometries, and I will be looking into this. (iv) I had the opportunity to tell David Gluck at length about my work with Alex Moreto on minimal non-zero heights in blocks. There is an open problem arising from this research to do with p-solvable groups, which David is very well equipped to tackle."

Fong: "I found myself fielding questions from Suzanne Dansk related to defect groups of blocks in classical groups, getting in turn references from Joe Chuang on extensions of derived equivalences between $SL_2(q)$ and a Sylow p-normalizer, catching up with the state of some of Britta Späth's current work on the inductive conjectures from Marc Cabanes, and puzzling still over a question by Jay Taylor related to Brauer trees in classical groups."

Geck: "I enormously enjoyed my stay at BIRS; the workshop was very inspiring for me. Besides getting a great overview on current developments and meeting and discussing with people whom I haven't seen for a while, I found particularly useful the problem session/discussion, in which a variety of problems (from major directions to important obstacles of a mere technical nature) were explained and possible ways discussed on how to overcome them. This certainly provided some fresh insights for me which – I hope – may even lead to some new research projects and collaborations."

Gluck: "G. Navarro mentioned a problem to me in August, 2013. At Banff, we had a useful discussion about this problem, and I am thinking about it now. Also at Banff, C. Eaton mentioned another interesting problem, which I may work on in the future."

Koshitani: "(1) I discussed interesting subjects in the modular representation theory of finite groups from global-local point of view with Charles Eaton. This could be new joint work. (2) I discussed at length with Caroline Lassueur a problem coming from endo-trivial modules which show up quite often in many important and interesting situations in representation theory of finite groups. Thanks to our stay at BIRS, we have been able to start our joint work and hopefully it would be completed soon. (3) I discussed integral representation theory of finite groups with Michael Geline, especially the so-called Knörr lattices. "

Külshammer: "I personally profited, in particular, from discussions with M. Geline, M. Linckelmann and A. Turull on vertices of simple modules and characters. R. Kessar, M. Linckelmann and I are in the course of preparation of a manuscript on this topic which will hopefully be available in a few weeks or perhaps months."

Lassueur: "As a young postdoc the workshop has been an excellent opportunity to meet and ask questions to the researchers working in my area and getting a better idea of who is doing what... I also used the week in Banff to work on a collaboration started two weeks earlier with Koshitani on the computation of Green correspondence for blocks with Klein-four defect group. We have proved a first result in Banff and could also put together a schedule for our future work. I also realized from discussions with J. Müller that the work I have recently done on computing Green correspondence of simple modules going upstairs for the sporadic groups could be a good starting point for future work on Broué's conjecture for some blocks of the sporadic groups in characteristic 5 or 7."

Müller: "In the aftermath of Robert Boltje's talk on p-permutation equivalences, together with him and Susanne Danz we started to discuss ideas how these might be examined computationally. This seems desirable to obtain a better understanding, as currently there is a lack of interesting substantial examples, in particular of p-permutation equivalences which do not already come from splendid derived equivalences. Actually, we will meet later this year again to pursue this."

Navarro: "I have started a new line of research with M. Linckelmann about characterizing nilpotent blocks, which is a possible continuation of one of my recent papers. Also, with my co-organizer P. H. Tiep, we have started a possible refinement of McKay in groups with nilpotent Hall subgroups. At the meeting, my student C. Vallejo gave her first important international talk, and she already received an invitation to visit Kaiserlautern with Gunter Malle for some months to join his team working on the McKay conjecture. I also spoke with Gluck about my conjecture with Gunter about characterizing nilpotent blocks."

Schaeffer Fry: "Following my talk, there were some very good recommendations and discussions, for example with G. Malle, G. Navarro, and J. Taylor, which may help my progress on one of my current problems. I also had the opportunity to speak with G. Malle and P.H. Tiep about possibilities for future directions for my research involving the inductive Alperin-McKay and Blockwise AWC conditions, as well as S. Koshitani about new ways to simplify some arguments in my earlier work on the inductive conditions, which will be quite helpful in my future research. The workshop has also produced the exciting possibility of visiting the institutions of others involved in the workshop in the coming year, namely M. Geline and G. Malle."

Srinivasan: "I had discussions with several participants, including Marc Cabanes, Joe Chuang, Olivier Dudas and Anton Evseev. One lecture in particular by Olivier Dudas on work he was doing with Gunter Malle was especially interesting for me and gave me insight into my research. Among new contacts I would specially mention Ben Elias, a young mathematician who is doing cutting edge research."

Tiep: "During my stay at BIRS, I was able to start a new research project with my collaborator Gabriel Navarro, as well as continue our collaboration on ongoing projects that aim to resolve some basic questions on representations of finite groups. Together with Gunter Malle and Frank Lübeck (and in discussion with Jean-Baptiste Gramain and Olivier Brunat), I had started another new research project with the aim to answer a question of R. M. Guralnick concerning representations of finite quasisimple groups. I also had discussions with other participants, including Marc Cabanes, Gerhard Hiss, Klaus Lux, Bhama Srinivasan, and Amanda Schaeffer Fry."

Turull: "While at Banff, I enjoyed productive mathematical discussion on these and related topics with a number of researchers, including Michael Geline, David Gluck, and Burkhard Külshammer. These and other informal exchanges affected a number of papers on these conjectures that I am in the process of writing, and suggested new promising avenues of research. It was an excellent and productive experience all around."

4.3 Conclusion

All the workshop's participants agreed that Banff lived up to its promises of a quiet, inspiring and very comfortable place to make mathematics. We are all very grateful to the BIRS for providing such excellent facility for discussing and doing mathematics, and hope to return some time in the future.

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