On the number of generators of a crystallographic group

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(P. Penninckx, A. Adem & N. Petrosyan & B. Putrycz)

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What are crystallographic groups?

- An n–dimensional crystallographic group Γ is a discrete cocompact subgroup of $\mathsf{Isom}(E^n)$ $(\subseteq \mathsf{GL}_{n+1}(\mathbb{R}))$.
- Γ fits in a short exact sequence

$$0 \to \mathbb{Z}^n \to \Gamma \to F \to 1$$

  where $\mathbb{Z}^n$ is the group of pure translations.
- $\mathbb{Z}^n$ is maximal abelian in Γ and F is finite.
  $\mathbb{Z}^n$ is a faithful F–module
- Γ has no non-trivial finite normal subgroups.

If Γ is torsion free
A Bieberbach group

$\rightsquigarrow \mathbb{R}^n/\Gamma$ is a (flat) manifold, otherwise orbifold.

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Why interest in the number of generators?

Let F be a finite group acting freely on a torus T^n, then $M = T^n/F$ is a manifold and

$$1 \to \pi_1(T^n) \cong \mathbb{Z}^n \to \pi_1(M) \to F \to 1$$

$\Rightarrow \pi_1(M)$ is Bieberbach

Info on generators of $\pi_1(M) \rightsquigarrow$ restrictions on possibilities for F.

One can replace T^n by any flat manifold.

One can also consider effective actions on flat orbifolds (crystallographic groups).

Results in case the holonomy is $\mathbb{Z}_p^h$

Theorem (D. – Penninckx, 2009)

Let p be a prime.
If Γ is an n-dimensional Bieberbach group with holonomy $\mathbb{Z}_p^h$,
then $\mathrm{rank}(\Gamma) \leq n$.

If p is odd, the same holds for crystallographic groups.

Corollary

If $\mathbb{Z}_p^h$ acts freely on T^n then $h \leq n$ and $h = n$ is possible.

(This was already proved by E. Yalçin in 2000.)

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Questions!!!!

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Is it true that any n-dimensional Bieberbach group can be generated by at most n elements?

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Is it true that any torsion-free polycyclic-by-finite group of Hirsch length n can be generated by at most n elements?

Question

Is there a bound on the number of generators a n–dimensional crystallographic group in terms of n? (Guess $2n$).

P.S. If we know the answer to the above question, then we also now it for polycyclic-by-finite groups without non-trivial finite normal subgroups.

Some evidence

Theorem (D. – Penninckx, 2009)

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- If $Z(\Gamma) \neq 0$, then $\Gamma/[\Gamma, \Gamma]$ is infinite, so there is a short exact sequence

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$\rightsquigarrow$ induction on dimension!

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  $\rightsquigarrow$ induction on dimension!
- E.g. in dim 6: 38746 Bieberbach groups, of which 27534 have holonomy $\mathbb{Z}_p^h$ and 33742 have non-trivial centre. "Only" 1399 groups to be checked (a few minutes).

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Case when holonomy is a general p–group

Theorem (D. – Adem, Petrosyan, Putrycz, 2012)

Let Γ be an n-dimensional crystallographic group s.t. its holonomy group F is a p-group. Then Γ can be generated by

$$a(n-\beta_1)/(p-1)+\beta_1$$

elements, where

- $a = 2$ *if* $p \leq 19$ *else* $a = 3$
- β_1 *is the rank of* $Z(\Gamma)$*, equals the torsion-free rank of* $\Gamma/[\Gamma,\Gamma]$.

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Proof: It suffices to consider $\Gamma = \mathbb{Z}^n \rtimes F$. So

- bound number of generators of F.
- bound number of generators of $\mathbb{Z}^n$ as an F–module.

Generators of F

Lemma

Let F be a nontrivial p–group and let M be an irreducible F–lattice.

$$\exists \alpha \in \mathbb{Z}_{\geq 0} : \operatorname{rank}(M) = p^{\alpha}(p-1)$$

and every subgroup of F is generated by at most p^{α} elements.

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Proposition

Let F be a nontrivial p–group and M a faithful F-lattice of rank n. Let $\beta_1 = \mathrm{rank}(M^F)$, then F is generated by $(n-\beta_1)/(p-1)$ elements.

Generators of M

Lemma

Let M be a $\mathbb{Z}_p$–module of rank n with $M^{\mathbb{Z}_p} = 0$, then as a $\mathbb{Z}_p$-module M is generated by $(a-1)n/(p-1)$ elements.

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Proposition

Let M be an F-lattice, $\beta_1 = \mathrm{rank}(M^F)$, then as a module M is generated by $(a-1)(n-\beta_1)/(p-1) + \beta_1$ elements.

Generators for the whole group

Let

$$1 \to M \to \Gamma \to F \to 1$$

be crystallographic with F a p-group.

generators for F: $(n - \beta_1)/(p-1)$

generators for M: $(a-1)(n-\beta_1)/(p-1) + \beta_1$

Together:

$$a(n-\beta_1)/(p-1) + \beta_1.$$