Non-Completely Reducible Subgroups of Exceptional Algebraic Groups

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November 17, 2014

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- Exceptional types G_2, F_4, E_6, E_7, E_8 $\rightsquigarrow$ more detail expected

G-complete reducibility

Definition (Serre)

A subgroup $X < G$ is called

- *G-reducible* if X lies in a proper parabolic subgroup P, and *G-irreducible* (G-irr) otherwise.
- *G-completely reducible* (G-cr) if $X < P \Rightarrow X < L < P$.

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[Holds for $p = 0$ or p 'big': Guralnick, Jantzen, McNinch, Serre, ...]

Non-G-cr subgroups

Fact (Liebeck+Seitz, Stewart)

If G is exceptional and p is good for G, then the isomorphism types of non-G-cr simple subgroups $X < G$ are

- $p = 5$, $X = A_1$, $G = E_6$ *or* E_7,
- $p = 7$, $X = A_1$ *or* G_2, $G = E_7$ *or* E_8,

(p good if $p > 3$, or $p > 5$ if $G = E_8$)

Theorem

Theorem (L., Thomas)

If G is an exceptional algebraic group in good characteristic, then the non-G-cr simple subgroups of G are known.

When $p = 5$:

- Countably ∞-many classes of subgroups A_1 in E_6 and E_7.

When $p = 7$:

- 2 classes of subgroups A_1 in E_7.
- 1 class of subgroups G_2 in E_7.
- Countably ∞-many classes of subgroups A_1 in E_8.
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Lots of extra information (normalisers, centralisers, action on G-modules, ...)

$P = Q \rtimes L$ is minimal among parabolics containing X

$\Rightarrow X < Q \rtimes X_0$ for some L-irreducible $X_0 < L$

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$$\{\phi(x)x : x \in X_0\}$$

$$\phi : X_0 \to V$$

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e.g. $p = 5$, $X = A_1 = SL(V)$

$S^5(V)$ not completely reducible $\Rightarrow$ non-cr embeddings into

$$A_5 < E_6 < E_7 < E_8$$

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- $\rightsquigarrow$ explicit formula for cocycles $X \to Q/[Q, Q]$
- $\rightsquigarrow$ algorithm for enumerating cocycles $X \to Q$.

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- Extension to bad characteristic ($G = E_6, E_7, E_8,\ p = 2, 3, 5$).

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- Applications to finite groups of Lie type.