Permutation groups where non-trivial elements have few fixed points

Algorithms for Linear Groups, Banff 2014

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In this talk G denotes a finite group that acts faithfully and transitively on a set Ω.

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We keep this hypothesis for the remainder of this talk.

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3. ... some four point stabiliser is non-trivial, but all five point stabilisers are trivial.

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In particular we want to completely classify all simple groups G that occur in these cases.

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- G has a subgroup of index at most 2 that is a Frobenius group.
- $|Z(G)| = 2$ and $G/Z(G)$ is a Frobenius group.
- G is soluble and the point stabilisers are metacyclic of odd order (more details).

Theorem, cont.

- The point stabilisers are metacyclic of odd order and there exists a prime power q such that G has a section isomorphic to $\mathrm{PSL}_2(q)$, to $\mathrm{Sz}(q)$ or to $\mathrm{PSL}_3(4)$.

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- The point stabilisers have twice odd order and G has a subgroup M of index 2 such that one of the previous two cases holds or M acts regularly on Ω.

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- The point stabilisers have twice odd order and G has a subgroup M of index 2 such that one of the previous two cases holds or M acts regularly on Ω.
- The point stabilisers have even order and G has a normal subgroup N of odd order such that $O^{2'}(G)/N$ is either a dihedral or semi-dihedral 2-group or there exists a prime power q such that it is isomorphic to $\mathrm{Sz}(q)$ or to a subgroup of $P\Gamma L_2(q)$ that contains $PSL_2(q)$.

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Then either G is isomorphic to $\mathrm{PSL}_3(4)$ or there exists a prime power q such that G is isomorphic to $\mathrm{PSL}_2(q)$ or to $\mathrm{Sz}(q)$.

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- $|G_\omega|$ is even and
  1. G has a normal 2-complement or
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  3. G (or an index 2 subgroup of G) has a strongly embedded subgroup.

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  4. G has a normal subgroup N which acts semi-regularly on Ω such that G/N is almost simple and G_ω is cyclic.

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1. $G \cong \mathcal{A}_5,\ \mathcal{A}_6,\ \mathcal{A}_7$ or $\mathcal{A}_8$ (with precise action explained).
2. $G \cong \mathrm{PSL}_2(7)$, $\mathrm{PSL}_2(11)$, $\mathrm{PSL}_3(4)$, $\mathrm{PSL}_4(3)$, $\mathrm{PSU}_4(3)$ or $\mathrm{PSL}_4(5)$ (with precise action described).

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5. $G \cong M_{22}$ acting on $2^7 \cdot 3^2 \cdot 5 \cdot 11$ points.

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Lemma

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Let us assume that this is false. We choose G to be a minimal counter-example and let $\alpha \in \Omega$.

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For all $1 \neq X \leq H$ we show that $N_G(X) \leq H$ and this implies that G is a Frobenius group, which is false.
Hence G_α has even order and we look at the 2-structure of G.

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3. The subgroup M acts transitively on Ω.
   Otherwise M has two orbits on Ω and we find some $y \in M$ of prime order p that fixes three points on Ω. This can only happen if all three fixed points of y are in one orbit, but then $p = 3$ and we have a contradiction.

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It follows that $\langle t \rangle \in \mathrm{Syl}_2(G)$ and that M has odd order, so it has order coprime to 6. Now M is a Frobenius group or it acts regularly on Ω.

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It follows that $\langle t \rangle \in \mathrm{Syl}_2(G)$ and that M has odd order, so it has order coprime to 6. Now M is a Frobenius group or it acts regularly on Ω.
In both cases we have a subgroup acting regularly, so its order is divisible by 3 (the number of fixed points of t). This is impossible.

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Again our objective is a general result about the structure of G together with a complete list of all simple groups that occur.

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We hope that our results might help to understand the automorphism groups of Riemann surfaces in situations where Weierstrass points cannot be found as fixed points of automorphisms.

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- Even if non-trivial elements have only few fixed points, these fixed points could still be Weierstrass points. What is going on?
- Can we construct explicit examples?
- Are there other applications?

Many thanks for your attention!