Units of Integral Group Rings, Banff, 2014

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G a finite group
$\mathbb{Z}G$ integral group ring
$\mathcal{U}(\mathbb{Z}G)$ group of invertible elements, a finitely presented group

PROBLEM (Sehgal-Ritter, 1989): describe constructively finitely many generators, in a generic way

PROBLEM (Sehgal): describe algebraic structure of $\mathcal{U}(\mathbb{Z}G)$

PROBLEM: Construct units

$$\mathbb{Z}G \subseteq \mathbb{Q}G = \oplus_i M_{n_i}(D_i),$$

each D_i a $\mathbb{Q}$-finite dimensional division algebra.

$$\mathbb{Z}G \subseteq \mathcal{O} = \oplus_i \mathbb{Z}Ge_i \subseteq \mathbb{Q}G$$

containment of ($\mathbb{Z}$-)orders (finitely generated $\mathbb{Z}$-modules that contain a $\mathbb{Q}$-basis of $\mathbb{Q}G$)
e_i primitive central idempotents

- unit groups of orders are commensurable
- If $\mathcal{O}$ is an order in a finite dimensional semisimple rational algebra A then $\langle \mathcal{O}^1, \mathcal{U}(Z(\mathcal{O})) \rangle$ has finite index in $\mathcal{U}(\mathcal{O})$. If, moreover, A is simple then $\mathcal{O}^1 \cap \mathcal{U}(Z(\mathcal{O}))$ is cyclic and finite.
- may take $\mathcal{O} = M_{n_1}(\mathcal{O}_1) \times M_{n_2}(\mathcal{O}_2) \times \cdots \times M_n(\mathcal{O}_n)$,

$\mathcal{O}^1$ reduced norm one elements, $\mathcal{O}_i$ order in D_i

- Solve problem first for central units
- Solve problem in each $\mathrm{SL}_{n_i}(\mathcal{O}_i)$
- Make constructions such that generators live in $\mathbb{Z}G$, or pull back into $\mathbb{Z}G$

Bass units

$$u_{k,m}(g) = (1 + g + g^2 + \cdots + g^{k-1})^m + \frac{1 - k^m}{|g|}\widetilde{g},$$

where $g \in G$ and k and m are positive integers such that $k^m \cong 1 \mod |g|$.

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Related to cyclotomic units

$$\frac{1-\zeta^k}{1-\zeta} = 1 + \zeta + \cdots + \zeta^{k-1}$$

Central Units

- G abelian: specific set of $u_{k,m_{k,C}}(a_C)$ that is a basis for a subgroup of finite index in $\mathcal{U}(\mathbb{Z}G)$
  Bass-Milnor, recent construction by JRV, without use of K-theory, main issue: construct units that only contribute to one simple factor
- G not abelian: for some classes of G (abelian-by-supersolvable) construction possible that is a product of conjugates of Bass units and that yields a basis for a subgroup of finite index in $Z(\mathcal{U}(\mathbb{Z}G))$.
  JORV 2013
- possible: because one gets a method to convert some units into central units (products of conjugates)

Let $g \in G$ of order not a divisor of 4 or 6 and let

$$\mathcal{N} : N_0 = \langle g \rangle \lhd N_1 \lhd N_2 \lhd \cdots \lhd N_m = G$$

be a subnormal series in G. For $u \in \mathcal{U}(\mathbb{Z}\langle g \rangle)$ define

$$c_0^{\mathcal{N}}(u) = u$$

and

$$c_i^{\mathcal{N}}(u) = \prod_{h \in T_i} c_{i-1}^{\mathcal{N}}(u)^h,$$

where T_i is a transversal for N_{i-1} in N_i, $i \geq 1$.

Bicyclic units

$g \in G$

$$\widetilde{g} = \sum_{i=0}^{|g|-1} g^i$$

Ritter-Sehgal introduced the bicyclic units

$$b(g, \widetilde{h}) = 1 + (1 - h)g\widetilde{h}$$

and

$$b(\widetilde{h}, g) = 1 + \widetilde{h}g(1 - h) \quad (g, h \in G),$$

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Related to

$$1 + (1 - e)\alpha e \quad \text{and} \quad 1 + e\alpha(1 - e),$$

with $e^2 = e$, ... elementary matrices

Theorem (Jespers-Leal, Ritter-Sehgal)

Suppose G does not have a non-abelian epimorphic image that is fixed point free and $\mathbb{Q}G$ does not have exceptional simple images. Then

group generated by Bass units and bicyclic units

is of finite index in $\mathcal{U}(\mathbb{Z}G)$.

Definition

A simple finite dimensional rational algebra is said to be *exceptional* if it is one of the following types:

1. a non-comm. div. algebra (not totally definite quat. alg.),
2. $M_2(\mathbb{Q})$,
3. $M_2(F)$ with F a quadratic imaginary extension of $\mathbb{Q}$,
4. $M_2\left(\left(\frac{a,b}{\mathbb{Q}}\right)\right)$, with a and b negative integers (i.e. $\left(\frac{a,b}{\mathbb{Q}}\right)$ is a totally definite quaternion algebra with center $\mathbb{Q}$).

- previous talk gives algorithms
- recent other algorithms via actions of discontinuous groups on hyperbolic spaces (JKR)
- case (4) fully described now, reduces to orders that are Euclidean
- generic constructions in orders in division algebras remain a black box

structure theorem

A rational algebra A is said to be of *Kleinian type* if it is either a number field, or a quaternion algebra over a number field F such that $\sigma(\mathcal{O}^1)$ is a discrete subgroup of $\mathrm{SL}_2(\mathbb{C})$ for some embedding σ of F in $\mathbb{C}$ and some order $\mathcal{O}$ of A.

Theorem (JPRRZ 2007)

Let $A = \prod_{i=1}^{k} A_i$ with each A_i a finite dimensional semisimple rational algebra. Let $\mathcal{O}$ be an order in A and for each A_i let $\mathcal{O}_i$ be an order in A_i. Then $\mathcal{U}(\mathcal{O})$ is virtually a direct product of free-by-free groups (respectively, of free groups) if and only if each $\mathcal{O}_i^1$ is virtually free-by-free (respectively, virtually free).

Theorem (JPRRZ 2007)

For a finite group G the following statements are equivalent.

(A) $\mathcal{U}(\mathbb{Z}G)$ *is virtually a direct product of free-by-free groups.*

(B) *For every simple quotient A of $\mathbb{Q}G$ and some (every) order $\mathcal{O}$ in A, $\mathcal{O}^1$ is virtually free-by-free.*

(D) *G is of Kleinian type, i.e. $\mathbb{Q}G$ is an algebra of Kleinian type.*

(E) *Every simple quotient of $\mathbb{Q}G$ is either a field, a totally definite quaternion algebra or $M_2(K)$, where K is either $\mathbb{Q}$, $\mathbb{Q}(i)$, $\mathbb{Q}(\sqrt{-2})$ or $\mathbb{Q}(\sqrt{-3})$.*

(F) *G is either abelian or an epimorphic image of $A \times H$, where A is abelian group of exponent 2,4 or 6 and H can in each case be well described.*

Problem: describe units in orders in finite dimensional rational division algebras