Multifractal Analysis: From Theory to Applications and Back

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1 Overview of the Field

Multifractals are mathematical objects characterized by unique and interesting scaling properties. Their study is approached commonly from one of the following interrelated perspectives:

- probabilistic modeling, statistics and stochastic processes,
- signal and image processing, and applications at large,
- functional analysis and geometric measure theory.

These perspectives will be referred to below as, respectively, **probability/statistics**, **applications** and **analysis**.

From the **probability/statistics** perspective, (random) processes $X = \{X(t)\}$ are *multifractal* when they exhibit the following scaling behavior:

$$E|T_X(a,t)|^q \sim C_q a^{\lambda(q)}. \tag{1}$$

Here, $T_X(a,t)$ is some *multiresolution quantity* related to X and corresponding to scale a and time (position) t. For example, $T_X(a,t)$ can be the increments

$$T_X(a,t) = X(t+a) - X(t) \tag{2}$$

or nth order increments, or the suitably normalized wavelet coefficients

$$T_X(a,t) = c_X(j,k) = \int X(t) 2^j \psi(2^j t - k) dt \tag{3}$$

corresponding to $a = 2^{-j}$, $t = k2^{-j}$, and where ψ is a (mother) wavelet function. Other, more sophisticated choices will be mentioned below. The quantity $E|T_X(a,t)|^q$ is the qth absolute moment of $T_X(a,t)$. The power q is such that $E|T_X(a,t)|^q < \infty$, though this is influenced not only by X but also by the choice of multiresolution quantity itself. $C_q > 0$ in (1) is a constant, and $\sim$ indicates the asymptotic equivalence as $a \to 0$, that is, at *small scales*. Exponents $\lambda(q)$ are referred to as *(theoretical) scaling exponents*. The focus is on nonlinear exponent functions $\lambda(q)$ which are thought to be associated with true multifractals. The linear case $\lambda(q) = c_1 q$, on the other hand, is thought to be associated with *monofractals*. The classical example of monofractals is that of *self-similar processes* for which there is a self-similarity exponent $H > 0$ such that the laws of the processes $X(ct)$ and $c^H X(t)$ are the same for all $c > 0$.

The class of self-similar processes has been studied quite extensively, including the processes such as Brownian motion, fractional Brownian motion, Rosenblatt processes, stable fractional motions and others (e.g. [49]). Truly multifractal processes include multiplicative cascades [38, 34], compound Poisson cascades [11], infinitely divisible cascades [10], non-scale invariant infinitely divisible cascades [19], multifractal random walks with respect to self-similar processes [8, 1], self-similar processes in multifractal time [39], random wavelet series [7].

From the **applications** perspective, expected values in (1) are replaced by sample averages, called the *structure function*,

$$S_n(q,a) = \frac{1}{n_a} \sum_{k=1}^{n_a} |T_X(a, t_k(a))|^q, \tag{4}$$

where n refers to the sample size of X and n_a to the number of multiresolution quantities $T_X(a, t_k(a))$ used in estimation at scale a. Since sample averages are natural estimators of expected values, it is expected in view of (1) that

$$S_n(q,a) \sim c_q a^{\zeta(q)}, \quad \text{as } a \to 0, n \to \infty. \tag{5}$$

Note that a different notation $\zeta(q)$ is used for the *(empirical) scaling exponents* in (5) (cf. (1)). This is not accidental. It turns out that for most multifractal models of interest,

$$\lambda(q) = \zeta(q), \quad \text{for } q_*^- < q < q_*^+, \tag{6}$$

but where, for example, $q_*^+ < q_c^+ = \sup\{q > 0 : E|T_X(a,t)|^q < \infty\}$, and the two functions $\lambda(q)$ and $\zeta(q)$ disagree for $q \in (q_*^+, q_c^+)$. With real data, $\zeta(q)$ is typically estimated through a regression of $\log S_n(q,a)$ against $\log a$ across a range of small scales a. Nonlinear estimated $\widehat{\zeta}(q)$ suggests that multifractals is a plausible model for the data at hand.

Data exhibiting multifractal properties arise and multifractal analysis is very popular in a wide range of applications, including those in physics and chemistry (fully developed turbulence, e.g. [17, 24, 38, 50], DNA sequences, e.g. [6]; diffusion-limited aggregation, e.g. [51]), earth and environmental sciences (topography, e.g. [20, 26]; earthquakes, e.g. [27]; river flows, e.g. [43]; rainfall, e.g. [25, 48]; cloud structure, e.g. [47]), image processing (natural images, e.g. [53, 18]; texture, e.g. [35, 57]), medicine and physiology (ECG, human heartbeat, e.g. [28, 52]; medical images, e.g. [44, 22]), computer science (network traffic, e.g. [46, 33, 54]), economics and finance (stock prices, e.g. [39, 9, 15]; volatility, e.g. [16]). This list is by far exhaustive. ScienceDirect alone gives over 4,000 results to the query "multifractal" - most of these in the applied literature.

From the **analysis** perspective, the focus is on the regularity analysis of deterministic functions. For a deterministic function $f = \{f(t)\}$, let

$$A_\alpha = \{t : \text{regularity of } f(t) \text{ at } t \text{ is } \alpha\}, \quad \alpha > 0. \tag{7}$$

Regularity of f in (7) is typically measured by using one of the multiresolution quantities $T_f(a,t)$ as in (2) and (3). One would say that f has regularity $\alpha = \alpha(t_0)$ at $t = t_0$ if α is the supremum of $h = h(t_0)$ for which

$$|T_f(a, t_0)| \le C a^h. \tag{8}$$

For multifractal functions f, the sets A_α, $\alpha > 0$, have a nontrivial structure in the sense that their Hausdorff dimension is positive for a range of α. In this case, one talks about the so-called *multifractal spectrum of singularities* of f defined as

$$d(\alpha) = \dim_H(A_\alpha), \tag{9}$$

where $\dim_H(A)$ indicates the Hausdorff dimension (see, e.g. [23]) of a set A. For deterministic functions, it is still meaningful to consider the corresponding structure function $S_n(q,a)$ in (4), and

$$\zeta(q) = \liminf_{a \to 0} \frac{\log S_n(q,q)}{\log a}. \tag{10}$$

It then turns out that, in many cases of interest, the functions d and ζ can be related through the Legendre transformation as

$$d(\alpha) = \inf_{q \neq 0} (1 - \zeta(q) + \alpha q) \tag{11}$$

(1 above is replaced by p in higher dimensions p).

The relation (11) is often referred to as *multifractal formalism* in the literature. The term "multifractal" refers to a range of exponents α characterizing the signal on various carrier sets A_α. On the functional analysis side, this approach based on regularity of deterministic functions has been advocated by a number of authors, most notably Jaffard [29, 30, 32].

2 Recent Developments and Open Problems

Exciting developments have recently been made in connection to all three directions listed in Section 1: **probability/statistics**, **applications** and **analysis**. Despite significant progress, however, numerous open questions remain to be addressed.

2.1 Recent Developments

For example, originating from the **analysis** direction, *wavelet leaders* [30, 31] have emerged as new and superior multiresolution quantities $T_X(a,t)$ in (1) to use in multifractal analysis. Wavelet leaders are defined as

$$T_X(a,t) = L_X(j,k) = \sup_{\lambda' \subset 3\lambda} |c_X(j',k')|. \tag{12}$$

Here, $\lambda = \lambda_{j,k} = [k2^{-j}, (k+1)2^{-j})$, $\lambda' = \lambda_{j',k'} = [k'2^{-j'}, (k'+1)2^{-j'})$ and $3\lambda = \lambda_{j,k-1} \cup \lambda_{j,k} \cup \lambda_{j,k+1}$. The supremum in (12) thus consists of the largest wavelet coefficient $c_X(j',k')$ computed at all finer scales $2^{-j'} \leq 2^{-j}$ within a narrow time neighborhood, $(k-1)2^{-j} \leq k'2^{-j'} < (k+2)2^{-j}$.

The basic idea behind wavelet leaders is that they allow to "look" into scaling exponents for negative values of q. In probabilistic terms, this results from taking the supremum in (12). With the supremum, the values of wavelet leaders that are close to zero become less likely and hence a higher number of negative moments become available. Even more surprisingly perhaps, wavelet leaders inherit the same scaling behavior (1) as when using differences or wavelet coefficients. Wavelet leaders have been used for multifractal analysis of real data in [55, 56, 58].

From the **probability/statistics** perspective, for example, significant developments are tied to recent progress in the probabilistic analysis of the so-called branching random walks (e.g. [4, 5]). Branching random walks are closely related to the classical constructions of multifractal measures such as multiplicative cascades. By using this connection, for example, the extremal properties of multiplicative cascades were derived in [40, 14], the multiplicative cascades at the so-called critical regime were studied in [12]. In other directions, probabilistic connections to certain PDEs can be found in [41]. Connections to quantum theory, the KPZ relation, appear in [13, 45]. Statistical inference questions are studied in [42, 36, 37].

Among recent **applications**, multifractal techniques involving wavelet leaders were extended to image analysis [57, 58]. An example concerns the classification of paintings [3]. Recent applications to medicine include those to heart rate variability [2] and fMRI [21]. A number of other applications are related to scaling at large.

2.2 Open Problems

At the **analysis** level, the multifractal formalism described in Section 1 above provides a potentially powerful tool to measure the multifractal spectrum from real-world data. Yet, a significant number of theoretical questions about its practical use remain open. What appropriate multiresolution quantities, function increments, wavelet coefficients, or more complicated quantities derived from wavelet coefficients such as wavelet leaders, should be used? What function spaces do they correspond to?

The multifractal formalism relies on a Legendre transform and thus on a concavity assumption. What can be done for data that do not fulfill such an assumption? As such, the multifractal formalism does not bring any additional information related to the nature of the singularities that exist in data. How should the formalism be modified to enable the detection of the local regularity of potentially oscillating nature (chirp-type), against the simpler case of non-oscillating (cusp-type) singularities? In several dimensions, how can anisotropy phenomena be analyzed?

At the **probabilisty/statistics** level, the gap between the deterministic definition of the multifractal spectrum and its application to each given sample path of stochastic processes remain to be formalized in a precise manner. Because real-world data are naturally envisaged as realizations of random models by practitioners, bridging that gap would pave the way towards crucial practical issues such as parameter estimation or hypothesis testing performance evaluation.

For example, what are the confidence intervals for given estimates? How can one test formally what model better fits data? Such issues could be addressed on a number of stochastic models designed to serve as reference and benchmark to compare applications to. Along similar lines, the (multi)fractal properties of data are often expected by practitioners to serve as a tool permitting to finely quantify or measure intricate statistical properties of data, such as strong dependencies or subtle departures of data from joint Gaussianity. These properties need to be understood in greater depth for benchmark multifractal processes.

At the **application** level, in many situations, because of the explosion of sensor designs and deployment, the data to analyze no longer consist of univariate signals (or functions). They could either take values in a d-dimensional space (a collection of time series is recorded jointly from one same system) or be indexed by a multiparameter (a field). Notably, there has been a growing interest to incorporate fractal and multifractal analysis into image processing toolboxes, with diverse applications ranging from medicine to satellite imagery. This change in the nature of data to be analyzed raised theoretical and practical issues that have so far been barely addressed.

For example, how should the multivariate multifractal spectrum be defined? How does the notion of anisotropy that naturally comes with images take its place in the multifractal analysis? How could one distinguish between an anisotropic but regular texture that has been superimposed to an isotropic fractal texture from a texture where the anisotropy is truly built-in its (multi)fractal properties? Images can exhibit boundaries (that split them into homogeneous subregions) that may themselves have fractal properties. Can they be distinguished from the fractal properties of the textures? Images may consist of the juxtaposition of fractal and regular patches of textures. How can they be identified?

At the modeling level, practitioners are often facing situations where the outcome of the analysis does not resemble the output of standard theoretical models. It would thus certainly be of interest to design a wide collection of deterministic and stochastic models, whose multifractal properties would be well-understood and could serve as a frame and guideline in the analysis of real-world data. For instance, developing models which incorporate oscillating singularities, or anisotropy, or which display non-concave or non-smooth multifractal spectra would be of great interest.

3 Presentation Highlights

The workshop had 18 talks spread over five days and 9 poster presentations in two sessions. One goal of the organizers was to have fewer talks along with unlimited opportunities for poster presentations, so that participants could have ample time for informal discussions. Another goal of the organizers was to gather researchers representing the three perspectives of multifractal analysis discussed above: **probability/statistics**, **applications** and **analysis**.

For example, the first day of the workshop already featured talks in the three directions. Murad Taqqu kicked off the workshop by presenting recent work on the Rosenblatt and related self-similar processes. The talk helped set stage for the workshop by reviewing the basic concepts behind monofractal processes. A highlight of the talk were several intriguing open problems about the marginal distribution of the Rosenblatt process. In a following talk on the same day, on the other hand, Julien Barral started by recalling the fundamental concepts behind multifractal analysis. Among the questions considered was that of the validity of multifractal formalism. An example included a finer behavior of multiplicative cascades in the regimes previously excluded from multifractal analysis. The first day of the workshop was closed with the talk of Herwig Wendt on wavelet leaders for multifractal analysis, touching upon not only their applied but also theoretical aspects.

Related to **probability/statistics**, two other talks were on anisotropic scaling of random fields. Gustavo Didier presented recent work on the structure of self-similar anisotropic, possibly vector random fields. Interesting points were made on the interplay between domain and range properties of such fields. One of the main messages was that a number of interesting issues arise even in the case of self-similarity (monofractality). Some of this was echoed in the talk by Beatrice Vedel. But in a significant step further, Beatrice Vedel also discussed ways of combining multifractality and anisotropy.

Paul Balança presented his recent work on oscillating singularities for Lévy processes. The existence of such singularities for any class of common stochastic processes has been a long standing problem. Paul Balança has made the first real progress in addressing this question, within the large family of Lévy processes. Edward Waymire spoke on the convergence of normalized multiplicative cascades in the so-called strong disorder. Using the results on branching random walks recently derived in the probability literature, not only the convergence could be proved but the limit could also be characterized in revealing ways.

In the poster sessions, the talk of Murad Taqqu was nicely complemented and extended by Shuyang Bai. Julien Hamonier's poster concerned estimation issues in the so-called linear multifractional stable motion. Nikolai Leonenko presented work on the construction of multifractal processes through products of geometric stationary processes, and on detection of multifractality under heavy tails. Peter Morter's poster considered the condensation phenomenon in the Kingman's model.

Other talks on **applications** were by: Shaun Lovejoy who spoke on scaling and multifractals in geophysics; Franklin Mendivil on fractal image processing; Philippe Ciuciu on multifractal modeling of fMRI signals; Ken Kyono on modeling heart rate variability; Alain Arneodo on scaling in genomics; and Patrick Flandrin on data-driven methods of scale invaraince analysis.

In the poster sessions, Patrice Abry presented his work on using multifractals for drawing classification. Joan Bruna advocated the use of scattering moments in multifractal analysis. The poster of Stephanie Randon de la Torre concerned applications to financial markets. Finally, Roberto Leonarduzzi considered the automatic selection of scaling range in multifractal analysis.

On the **analysis** side, Jun Kigami spoke on the multifractal structures arising with diffusions in inhomogeneous media, with a surprising role payed by a metric chosen in the multifrcatal analysis. Ka-sing Lau's talk focused on spectral properties of self-similar sets and measures, describing the recent progress made in this challenging research direction. Stephane Seuret spoke on p–exponents and p–multifractal spectrum of some lacunary Fourier series, with fascinating connections to number theory, harmonic analysis and dynamical systems.

Yang Wang revisited the celebrated Cantor set by raising surprising open questions and providing partial answers. Michel Zinsmeister spoke on the multifractality of whole-plane SLE. Finally, in a poster session, Céline Esser revisited the so-called S^ν spaces with wavelet leaders to detect non concave and non increasing spectra.

4 Scientific Progress Made and Outcome of the Meeting

The organized workshop has met its main objective of bringing together experts in fractals and multifractals from three different areas of research:

- probabilistic modeling, statistics and stochastic processes,
- signal and image processing, and applications at large,
- functional analysis and geometric measure theory.

Since workshops gathering researchers from these three communities are rare, the meeting was unique in this sense.

In bringing together these research areas, cross-fertilization between communities was fostered, easing transfers of theoretical results to real applications, as well as relevant applied questions to theoretical formalizations. The meeting provided an excellent overview of the field, from the three perspectives described above. It generated, often lively, discussions on open and crucial issues in multifractal analysis. The meeting also served as the starting point of new collaborations between researchers with different cultures and backgrounds in order to go towards the resolution of these issues.

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