Crystal operators and flag Gromov-Witten invariants

Anne Schilling (UC Davis)
joint with Jennifer Morse (Drexel)
Banff, June 17, 2014

- Littlewood-Richardson template
- Variations
- k-Schur functions and dual k-Schur functions
- Crystal operators on affine factorizations

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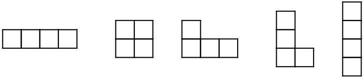
Variation 1: Littlewood-Richardson coefficients $c^{\nu}_{\lambda\mu}$

Indexed by partitions: 

- Tensor product multiplicities

$$V(\lambda) \otimes V(\mu) = \bigoplus_{\nu} c^{\nu}_{\lambda\mu} V(\nu)$$

- Symmetric function coefficients

$$s_\lambda \, s_\mu = \sum_{\nu} c^{\nu}_{\lambda\mu} \, s_\nu \qquad \text{and} \qquad s_{\nu/\lambda} = \sum_{\mu} c^{\nu}_{\lambda\mu} \, s_\mu$$

- Intersections in the Grassmannian

$$c^{\nu}_{\lambda\mu} = X_\lambda \cap X_\mu \cap X_{\hat{\nu}}$$

- Cohomology of the Grassmannian structure constants

$$\sigma_\lambda \cup \sigma_\mu = \sum_{\nu \subset rect} c^{\nu}_{\lambda\mu} \, \sigma_\nu$$

Combinatorial description

Littlewood–Richardson rule

$c^{\nu}_{\lambda\mu} = \#$ skew tableaux t of shape ν/λ and weight μ such that $\mathrm{row}(t)$ is a reverse lattice word.

Example

$$s_{21}\, s_{21} = \cdots + ?s_{321} + \cdots$$

211

121

112

$\Rightarrow c^{321}_{21,21} = 2$

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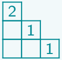

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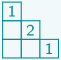

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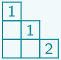

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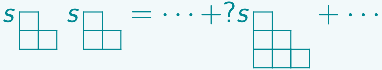

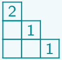

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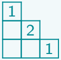

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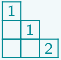

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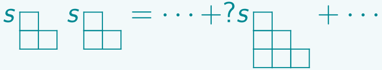

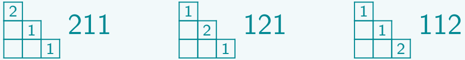

$\Rightarrow c^{321}_{21,21} = 2$

Gordon James (1987) on the Littlewood-Richardson rule:

> *"Unfortunately the Littlewood-Richardson rule is much harder to prove than was at first suspected. The author was once told that the Littlewood-Richardson rule helped to get men on the moon but was not proved until after they got there."*

Crystal graph

Action of crystal operators e_i, f_i, s_i on tableaux:

1. Consider letters i and $i+1$ in row reading word of the tableau
2. Successively "bracket" pairs of the form $(i+1, i)$
3. Left with word of the form $i^r(i+1)^s$

$$e_i(i^r(i+1)^s) = \begin{cases} i^{r+1}(i+1)^{s-1} & \text{if } s > 0 \\ 0 & \text{else} \end{cases}$$

$$f_i(i^r(i+1)^s) = \begin{cases} i^{r-1}(i+1)^{s+1} & \text{if } r > 0 \\ 0 & \text{else} \end{cases}$$

$$s_i(i^r(i+1)^s) = i^s(i+1)^r$$

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Crystal reformulation

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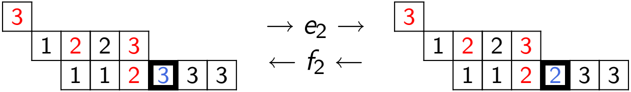
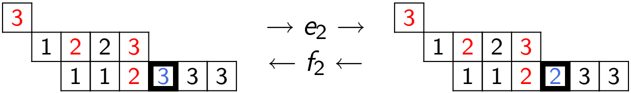


e_2: change leftmost unpaired 3 into 2

f_2: change rightmost unpaired 2 into 3

Crystal reformulation



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Theorem

b where all $e_i(b) = 0$ (highest weight)

$\leftrightarrow$ *connected component*

$\leftrightarrow$ *irreducible*

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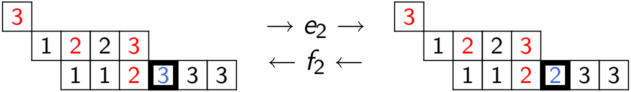

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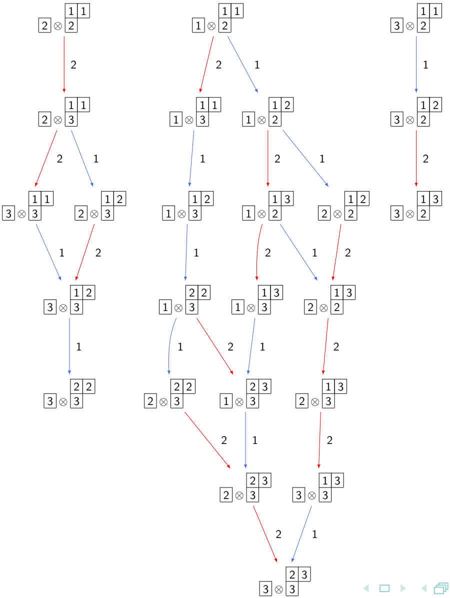
Reformulation of LR rule

$c^{\nu}_{\lambda\mu}$ counts tableaux of shape ν/λ and weight μ which are highest weight.

Decomposition



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- k-**Schur functions and dual** k-**Schur functions**
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Variation 2: c_{uv}^w

The set $\mathbb{F}_n$ of complete flags:

$$0 = W_0 \subset W_1 \subset \cdots \subset W_n = \mathbb{C}^n$$

subvarieties indexed by permutations of S_n

Intersections in the flag variety

Count points in the intersection $c_{uv}^w = X_u \cap X_v \cap X_{w_0 w}$

Structure constants in cohomology of the flag variety

$$\sigma_u \cup \sigma_v = \sum_{w \in S_n} c_{uv}^w \, \sigma_w$$

Schubert polynomial coefficients

$$\mathfrak{S}_u \, \mathfrak{S}_v = \sum_w c_{uv}^w \, \mathfrak{S}_w$$

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Variations 1 and 2 quantized

Grassmannian | Flags

Gromov-Witten invariants
Quantum cohomology

count rational curves of degree d that meet $X_\lambda, X_\mu, X_{\hat{\nu}}$

count equivalence classes of rational curves of multidegree d in $\mathbb{F}_n$

$$\sigma_\lambda *_q \sigma_\mu = \sum_{\nu \subset rect} q^d \, N_{\lambda\mu}^{\nu} \, \sigma_\nu$$

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Polynomial coefficients modulo an ideal

Ring of symmetric functions

$\mathbb{Z}[x_1, \ldots, x_n; q_1, \ldots, q_{n-1}]$

Schur functions

quantum Schubert polynomials

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Modulo an ideal is non-trivial

$$s_\lambda \, s_\mu \; = \; \sum_{\nu \subset rect} c^\nu_{\lambda\mu} \, s_\nu + \sum_{\nu \not\subset rect} c^\nu_{\lambda\mu} \, s_\nu$$

$$\Lambda \otimes \mathbb{Z}[q] \twoheadrightarrow QH^*(Gr_{a,n})$$

$$s_\lambda \; \mapsto \; \begin{cases} \sigma_\lambda & \text{when} \quad \lambda \subset \textit{rectangle} \\ \pm q^* \sigma_{\tilde{\lambda}} & \text{when} \quad \lambda \not\subset \textit{rectangle} \end{cases}$$

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- Originally an empirical study [Lascoux, Lapointe, Morse], for $\lambda_1 \leq k$,

$$H_\lambda(x; q, t) = \sum_{\mu_1 \leq k} K_{\lambda\mu}(q, t)\, A_\mu^{(k)}(x; t)\,,$$

  where $K_{\lambda\mu}(q, t) \in \mathbb{N}[t]$.

- Various definitions: one family of functions $\{s_\mu^{(k)}\}_{\mu_1 \leq k}$ defined in terms of a k-Pieri rule is conjectured to satisfy $A_\mu^{(k)}(x; 1) = s_\mu^{(k)}$

- $\{s_\mu^{(k)}\}_{\mu_1 \leq k}$ basis for $\Lambda_{(k)} = \mathbb{Z}[h_1, \ldots, h_k]$

- $s_\mu^{(big)} = s_\mu$

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- $\mathcal{F}_\lambda(x) \in \Lambda^{(k)} = \Lambda/\langle m_\lambda \mid \lambda_1 > k\rangle$ dual to k-Schur functions under Hall inner product $\langle \cdot, \cdot \rangle \colon \Lambda_{(k)} \times \Lambda^{(k)} \to \mathbb{Q}$

- Dual k-Schur functions are special cases of affine Stanley symmetric functions F_w when $w \in \tilde{S}_{k+1}$ is an affine Grassmannian permutation.

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Variation 1q: quantized $c_{\lambda\mu}^{\nu}$

Wess-Zumino-Witten model of Verlinde algebra

Gromov-Witten invariants of the Grassmannian

$$\sigma_\lambda *_q \sigma_\mu = \sum_{\substack{\nu \subset rect \\ |\nu|=|\lambda|+|\mu|-dn}} q^d N_{\lambda\mu}^{\nu} \sigma_\nu$$

Symmetric function coefficients

- Schur coefficients in product of Schur functions modulo an ideal
- k-Schur coefficients in a product of k-Schur functions

$$s_\lambda^{(k)} s_\mu^{(k)} = \sum_{\hat{\nu}=(a^*,\nu\subset rect)} N_{\lambda\mu}^{\nu} s_{\hat{\nu}}^{(k)} + \sum_{\hat{\nu}\neq(a^*,\nu\subset rect)} c_{\lambda\mu}^{\hat{\nu}} s_{\hat{\nu}}^{(k)}$$

Computation in Λ

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Computation in Λ

Variation 2q: Flag Gromov–Witten invariants

Affine Grassmannian

$\tilde{G}r = SL(n, \mathbb{C}((t)))/SL(n, \mathbb{C}[[t]])$ $\quad n = k + 1$

homology of affine Grassmannian $\twoheadrightarrow$ quantum cohomology of Grassm.

$\downarrow$

quantum cohomology of flags

Product of k-Schurs **Flag Gromov-Wittens**

$s_\lambda^{(k)} s_\mu^{(k)} = \sum_\nu C_{\lambda\mu\nu} s_\nu^{(k)}$ $\quad \sigma_u *_q \sigma_v = \sum_w \sum_d q^d \langle u, v, w \rangle_d \sigma_{w_0 w}$

k-bounded partitions permutations of S_{k+1}

Theorem (Morse-Lapointe)

Precise relation between $C_{\lambda\mu\nu}$ and $\langle u, v, w \rangle_d$ (up to relabeling).

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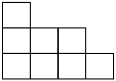
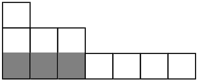
Indexing sets

k-bounded partition $\qquad$ $k+1$-core $\qquad$ $(k=4)$

$\lambda =$  $\leftrightarrow$  $=$

Action of affine symmetric group on cores:

$$s_i \tau = \tau + \begin{cases} \text{all boxes of residue } i \text{ added} \\ \text{all boxes of residue } i \text{ removed} \\ \text{nothing} \end{cases}$$

$\emptyset \xrightarrow{s_0}$

$\xrightarrow{s_4 s_3 s_2 s_1}$

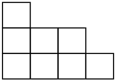
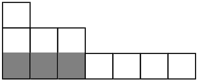
$\xrightarrow{s_1 s_0}$

$\xrightarrow{s_3}$

Affine Grassmannian element in $\tilde{S}_{k+1}/S_{k+1}$: $\quad \tilde{w}_\lambda = s_3 s_1 s_0 s_4 s_3 s_2 s_1 s_0$

Indexing sets

k-bounded partition $\qquad$ $k+1$-core $\qquad$ $(k = 4)$

λ =  $\leftrightarrow$  =

3						
4	0	1				
0	1	2	3	4	0	1

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$$\emptyset \xrightarrow{s_0} \begin{array}{|c|}\hline 0 \\ \hline\end{array} \xrightarrow{s_4 s_3 s_2 s_1} \begin{array}{ccccc} 4 & & & & \\ 0 & 1 & 2 & 3 & 4 \end{array} \xrightarrow{s_1 s_0} \begin{array}{ccccccc} 4 & 0 & 1 & & & & \\ 0 & 1 & 2 & 3 & 4 & 0 & 1 \end{array} \xrightarrow{s_3} \begin{array}{ccccccc} 3 & & & & & & \\ 4 & 0 & 1 & & & & \\ 0 & 1 & 2 & 3 & 4 & 0 & 1 \end{array}$$

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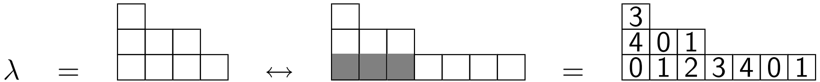

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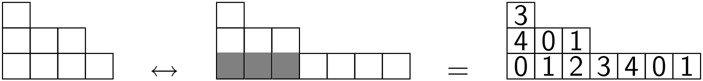

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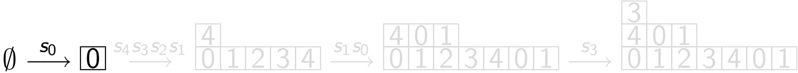

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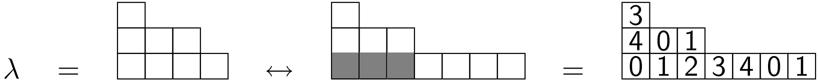

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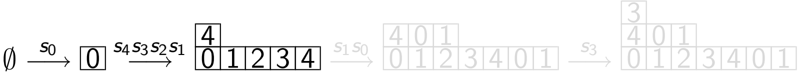

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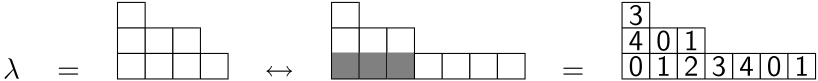

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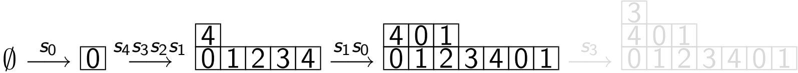

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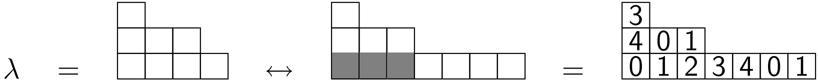

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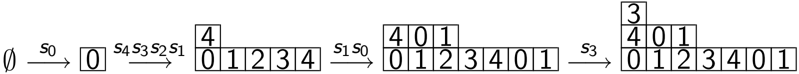

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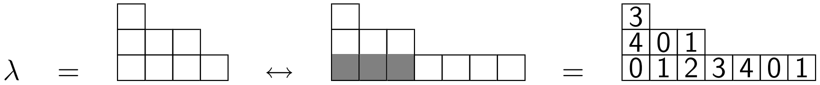

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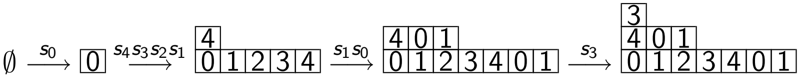

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Affine symmetric group

Affine symmetric group $\tilde{S}_n$

$\langle s_0, s_1, \ldots, s_{n-1} \rangle$ where
$$s_i s_j = s_j s_i$$
$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \qquad \text{(all indices } \mod n)$$
$$s_i^2 = 1$$

Example

For $n = 3$, $\quad s_1 s_2 s_1 s_0 = s_2 s_1 s_2 s_0$

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Affine Grassmannian permutations

All reduced words end in s_0

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Affine Grassmannian permutations

All reduced words end in s_0

Affine horizontal strips and Pieri rule

Schur function Pieri rule

$$h_r s_\lambda = \sum_{\substack{\nu \\ \nu/\lambda \text{ horizontal } r\text{-strip}}} s_\nu$$

k-Schur function Pieri rule

$$h_r s_\lambda^{(k)} = \sum_{\substack{\nu \\ \nu/\lambda \text{ weak horizontal } r\text{-strip}}} s_\nu^{(k)}$$

ν/λ is weak horizontal r-strip if $\tilde{w}_\nu \tilde{w}_\lambda^{-1}$ is cyclically decreasing of length r.

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Cyclically decreasing permutation

Definition

$\tilde{w} \in \tilde{S}_n$ is cyclically decreasing if every reduced word has no $j-1$ preceeding $j \pmod{n}$.

Remark

In particular, every letter in the reduced word appears at most once.

Example

For $n = 4$, cyclically decreasing: $\tilde{w} = s_1 s_0 s_3$ and $\tilde{w} = s_3 s_1$
not cyclically decreasing $\tilde{w} = s_3 s_1 s_0$

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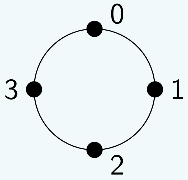
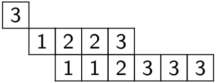


k-tableaux or affine factorizations

Schur case

tableau $\leftrightarrow$ sequence of horizontal strips



$\leftrightarrow$

k-Schur case

horizontal strip $\leftrightarrow$ cyclically decreasing element

Definition

A k-tableau or affine factorization of shape λ and weight α is a factorization of $\tilde{w}_\lambda = v^r \cdots v^1$ such that:

- $\ell(\tilde{w}_\lambda) = |\alpha|$
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k-tableaux or affine factorizations

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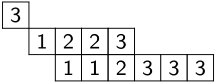

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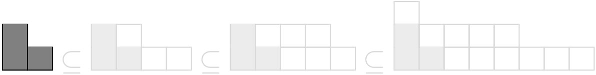

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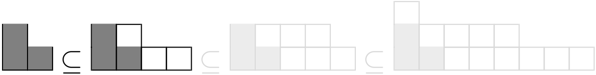
Schur case

tableau $\leftrightarrow$ sequence of horizontal strips

3							
	1	2	2	3			
		1	1	2	3	3	3

$\leftrightarrow$



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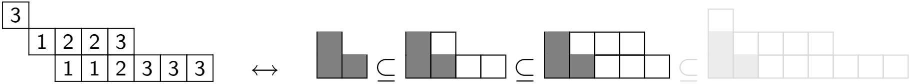

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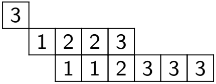

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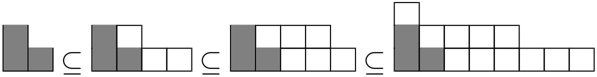

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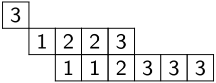

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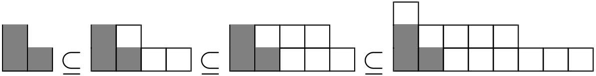

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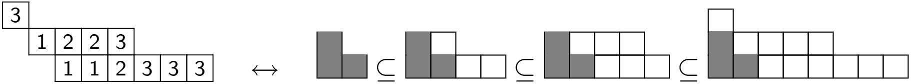

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Affine factorizations of $\tilde{w}_\lambda = s_3 s_2 s_3 s_1 s_0 = s_2 s_3 s_2 s_1 s_0 \in \tilde{S}_4$

with weight $\alpha = (21^3)$ $\{(s_3)(s_2)(s_3)(s_1 s_0), (s_2)(s_3)(s_2)(s_1 s_0)\}$

with weight $\alpha = (122)$ $\{(s_3 s_2)(s_3 s_1)(s_0)\}$

k-tableaux or affine factorizations (continued)

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Crystal operators and flag Gromov-Witten invariants

- **Littlewood-Richardson template**

- **Variations**

- ***k*-Schur functions and dual *k*-Schur functions**

- **Crystal operators on affine factorizations**

- **Littlewood-Richardson template**
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Schur times k-Schur

k-Schur coefficients in $s_\mu \, s_{\tilde{\nu}}^{(k)}$ include

- all fusion coefficients
- coefficients in Schur times a Schubert polynomial
- Gromov-Witten invariants for flags $\langle u, v, w \rangle_d$ where u has one descent

Strategy

- Affine Stanley symmetric functions are generating functions of affine factorizations
$$F_w = \sum_{v^r \cdots v^1 = w} x_1^{\ell(v^1)} \cdots x_r^{\ell(v^r)}$$

- Affine Stanley is skew dual k-Schur always (new observation!)
$$F_w = \mathcal{F}_{\nu/\lambda}$$

- Expansion gives k-Schur coefficients
$$\mathcal{F}_{\nu/\lambda} = \sum_{\mu} c^{w_\nu,k}_{w_\lambda,w_\mu} \mathcal{F}_\mu$$

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Crystal on affine factorizations yields combinatorial expression for Schur coefficients!

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Strategy

- Affine Stanley symmetric functions are generating functions of affine factorizations

$$F_w = \sum_{v^r \cdots v^1 = w} x_1^{\ell(v^1)} \cdots x_r^{\ell(v^r)}$$

- Affine Stanley is skew dual k-Schur always (new observation!)

$$F_w = \mathcal{F}_{\nu/\lambda}$$

- Expansion gives k-Schur coefficients

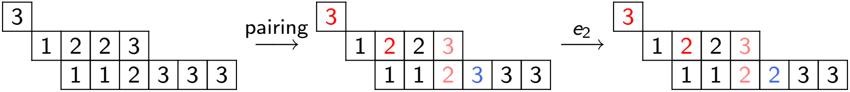
$$\mathcal{F}_{\nu/\lambda} = \sum_{\mu} c^{w_\nu, k}_{w_\lambda, w_\mu} \mathcal{F}_\mu$$

- Reduces to Schur expansion when $\mu \subset (r^{n-r})$.

Crystal on affine factorizations yields combinatorial expression for Schur coefficients!

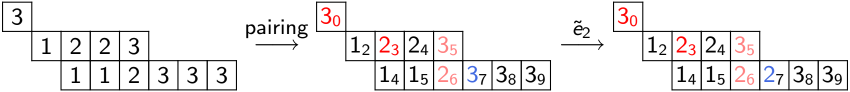
Crystal operators on affine factorizations

Recall e_i pairing and action:



Crystal operators on affine factorizations

Label cells diagonally



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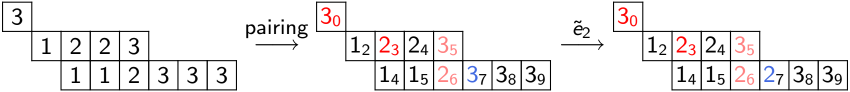

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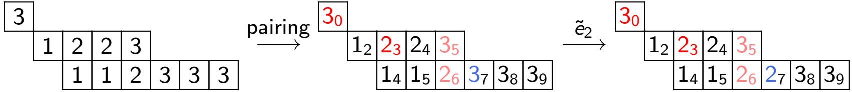

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from big to small:
pair $x \in$ 3's with smallest $y \in$ 2's that is bigger than x

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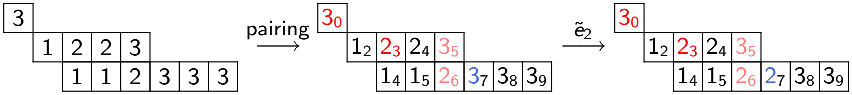

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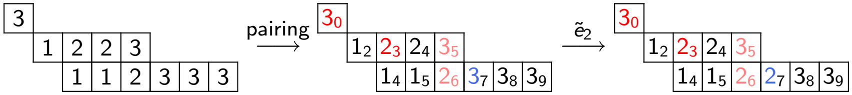

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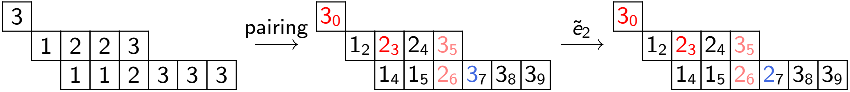

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$$(9\,8\,7\,5\,4\,3\,0)(8\,5\,4\,1\,0) \to (9\,8\,7\,4\,3\,0)(8\,5\,4\,3\,1\,0)$$

Main Results (with Morse)

Definition

Fix $w \in \langle s_0, \ldots, \widehat{s_x}, \ldots, s_{n-1} \rangle = S_{\hat{x}}$.

Graph $B(w)$

1. vertices are affine factorizations of w
2. edges are imposed and colored by $\tilde{f}_i$, $\tilde{e}_i$
3. highest weights are vertices with no unpaired entries

Theorem

$B(w)$ is a crystal graph of type A_ℓ

Proof

Checking Stembridge local axioms

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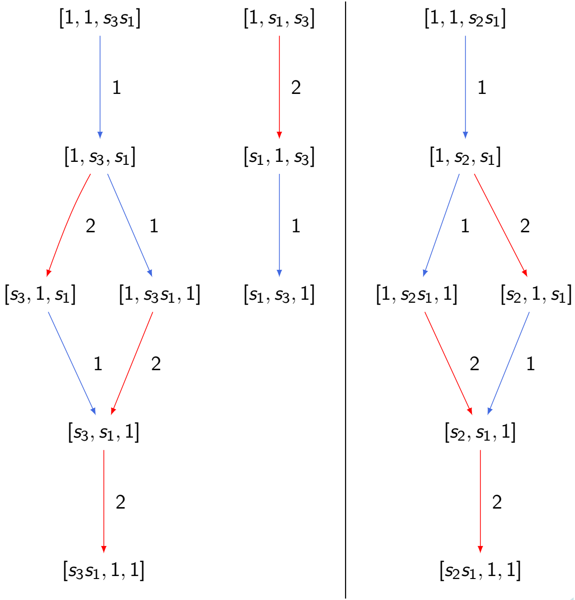
[1, 1, s3s1]
1
[1, s3, s1]
2
1
[s3, 1, s1]
[1, s3s1, 1]
1
2
[s3, s1, 1]
2
[s3s1, 1, 1]
[1, s1, s3]
2
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1
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Main Results (with Morse)

Theorem

For partition $\mu \subseteq (a^{n-a})$ and affine Grassmannian $\tilde{v}$, let

$$s_\mu \, s_{\tilde{v}}^{(k)} = \sum_{\tilde{w}} c_{\mu \tilde{v}}^{\tilde{w}} \, s_{\tilde{w}}^{(k)} \, .$$

If $\tilde{w}\tilde{v}^{-1} \in \langle s_0, \ldots, s_{\hat{x}}, \ldots, s_{n-1} \rangle$,
$c_{\mu,\tilde{v}}^{\tilde{w}} = \#$ of affine factorizations of $\tilde{w}\tilde{v}^{-1}$ with weight μ killed by all $\tilde{e}_i$.

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Corollaries

Corollary

Schubert polynomial expansion of $s_\lambda \mathfrak{S}_w$ for any $w \in S_n$ and partition λ where $|\lambda^c| < n$.

Corollary

Fusion rules $N_{\lambda\mu}^{\nu}$ for any λ, μ and ν such that

- ν/μ has a cut-point
- or λ satisfies $|\lambda^c| < n$.

Corollary

Gromov-Witten invariants for flags $\langle u, v, w \rangle_d$ when u has one descent and $v_r w_{R_r} w^{-1} \in S_{\hat{x}}$
(v_r is v shifted by r; w_{R_r} element obtained from rth k-rectangle)

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Quantum Flag

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Affine Stanley symmetric functions

indexed by affine permutations

$$F_w = \sum_{v^r \cdots v^1 = w} x_1^{\ell(v^1)} \cdots x_r^{\ell(v^r)}$$

Affine factorizations of w

- w is a product of affine permutations $v^r \cdots v^1$
- each v^i is cyclically decreasing
- $\ell(w) = \ell(v^1) + \cdots + \ell(v^r)$

some affine factorizations of $w = s_3 s_2 s_3 s_1 s_0 \in \tilde{S}_4$

$$(s_3)(s_2)(s_3)(s_1 s_0) \longrightarrow x_1^2 x_2 x_3 x_4$$

$$(s_2)(s_3)(s_2)(s_1 s_0) \longrightarrow x_1^2 x_2 x_3 x_4$$

$$(s_2)(s_3)(s_2 s_1 s_0) \longrightarrow x_1^3 x_2 x_3$$

$(s_2)(s_3 s_2 s_1 s_0)$ is BAD

Schur expansion

Fix $w \in \langle s_0, \ldots, s_{\hat{x}}, \ldots, s_{n-1} \rangle \subset \tilde{S}_n$

Theorem (with Morse)

$$F_w = \sum_{\lambda} a_{w\lambda}\, s_\lambda$$

$a_{w\lambda}$ *counts highest weights* $v^r \cdots v^1$ *of* $B(w)$ *with* $(\ell(v^1), \ldots, \ell(v^r)) = \lambda$

In $\tilde{S}_4$ (where $0 > 3$):

$\Longrightarrow\ F_{s_2 s_0} = s_2 + s_{1,1}$

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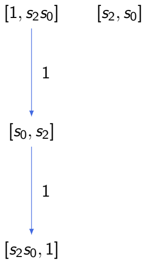

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Generalized Specht modules

D_w diagram of permutation

Theorem (Kraśkiewicz 1995, Reiner-Shimozono 1995)

$w \in S_n$, $p = \ell(w)$, $\lambda \vdash p$

Then $a_{w\lambda}$ is the multiplicity of the irreducible S_n-representation S^λ in the generalized Specht module M_{D_w}.

crystal interpretation for non-skew shapes!

Theorem (with Morse)

For any permutation $\tilde{w} \in S_{\hat{x}} \subset \tilde{S}_n$, the crystal isomorphism

$$B(\tilde{w}) \cong \bigcup_\lambda B(\lambda)$$

is explicitly given by the Edelman-Greene insertion $\varphi^Q_{\mathrm{EG}}(v^\ell \cdots v^1) = Q$:

$$\varphi^Q_{\mathrm{EG}} \circ \tilde{e}_i = \tilde{e}_i \circ \varphi^Q_{\mathrm{EG}}$$

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Further/future Work

- Positroids of Knutson-Lam-Speyer indexed by bounded affine permutations
- Gromov-Witten invariants
  Closer study of crystal structure on affine factorizations and crystal operators on dual k-tableaux
- t-analogue of k-Schur functions and relation to energy on KR crystals (charge plus offset) $\Rightarrow$ generalization to other types
- Other types
- K-theory analogue of the crystal operators

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