Constraining gravitational interactions in the M theory effective action

Anirban Basu
HRI, Allahabad

January 17, 2014
Outline of the talk

- Introduction and motivation
  - A class of local M theory interactions
  - The spacetime structure of the interactions in M/string theory
  - Constraints from supersymmetry and S–duality in IIB
  - The $D^{12} R^4$ interaction
  - Conclusions
Introduction and motivation
A class of local M theory interactions
The spacetime structure of the interactions
Constraints from supersymmetry and S–duality in IIB
The $D^{12}R^4$ interaction
Conclusions

Outline of the talk

- Introduction and motivation
- A class of local M theory interactions
  - The spacetime structure of the interactions in M/string theory
  - Constraints from supersymmetry and S–duality in IIB
  - The $D^{12}R^4$ interaction
- Conclusions
Outline of the talk

- Introduction and motivation
- A class of local M theory interactions
- The spacetime structure of the interactions in M/string theory
- Constraints from supersymmetry and S–duality in IIB
- The $D^{12} R^4$ interaction
- Conclusions
Outline of the talk

- Introduction and motivation
- A class of local M theory interactions
- The spacetime structure of the interactions in M/string theory
- Constraints from supersymmetry and S–duality in IIB
  - The $D^{12}R^4$ interaction
- Conclusions
Outline of the talk

- Introduction and motivation
- A class of local M theory interactions
- The spacetime structure of the interactions in M/string theory
- Constraints from supersymmetry and S–duality in IIB
- The $D^{12} \mathcal{R}^4$ interaction
- Conclusions
Outline of the talk

- Introduction and motivation
- A class of local M theory interactions
- The spacetime structure of the interactions in M/string theory
- Constraints from supersymmetry and S–duality in IIB
- The $D^{12}R^4$ interaction
- Conclusions
It is important to understand the effective action of string theory/M theory in various backgrounds.

The effective action encodes important information about the various duality symmetries of the theory, which allows us to calculate various perturbative as well as non-perturbative effects. Every term in the effective action encodes non-trivial information about the S matrices of the theory.

This effective action leads to duality covariant equations of motion.
It is important to understand the effective action of string theory/M theory in various backgrounds.

The effective action encodes important information about the various duality symmetries of the theory, which allows us to calculate various perturbative as well as non–perturbative effects. Every term in the effective action encodes non–trivial information about the S matrices of the theory.

This effective action leads to duality covariant equations of motion.
It is important to understand the effective action of string theory/M theory in various backgrounds.

The effective action encodes important information about the various duality symmetries of the theory, which allows us to calculate various perturbative as well as non–perturbative effects. Every term in the effective action encodes non–trivial information about the S matrices of the theory.

This effective action leads to duality covariant equations of motion.
Calculating the effective action is difficult in general, but certain terms in the effective action in maximally supersymmetric theories can be calculated exactly, these are BPS protected.

The non–BPS terms are much more difficult to determine. I shall discuss a particular case in detail where string theory gives some information about such a term in the M theory effective action.
Calculating the effective action is difficult in general, but certain terms in the effective action in maximally supersymmetric theories can be calculated exactly, these are BPS protected.

The non–BPS terms are much more difficult to determine. I shall discuss a particular case in detail where string theory gives some information about such a term in the M theory effective action.
Our aim is to look at the local, purely gravitational interactions in the M theory effective action in 11 dimensional flat spacetime, perhaps the simplest case to study.

This theory has 32 supersymmetries, which should constrain its effective action, like other theories with such a large amount of supersymmetry.
Our aim is to look at the local, purely gravitational interactions in the M theory effective action in 11 dimensional flat spacetime, perhaps the simplest case to study.

This theory has 32 supersymmetries, which should constrain its effective action, like other theories with such a large amount of supersymmetry.
Let us consider local, purely gravitational interactions in the M theory effective action of the form

\[ S = l_{11}^{2k-3} \int d^{11}x \sqrt{-G}D^{2k}R^4. \]

Compactify on a circle of (dimensionless) radius \( R_{11} \) such that

\[ l_{11} = e^{\phi_A/3}l_s, \quad R_{11}^3 = e^{2\phi_A}. \]

The length element is given by

\[ ds^2 = G_{MN}dx^Mdx^N = g_{\mu\nu}dx^\mu dx^\nu + R_{11}^2(dx^{11} - C_\mu dx^\mu)^2. \]
Let us consider local, purely gravitational interactions in the M theory effective action of the form

\[ S = l_{11}^{2k-3} \int d^{11}x \sqrt{-g} D^{2k} R^4. \]

Compactify on a circle of (dimensionless) radius \( R_{11} \) such that

\[ l_{11} = e^{\phi_A/3} l_s, \quad R_{11}^3 = e^{2\phi_A}. \]

The length element is given by

\[ ds^2 = G_{MN} dx^M dx^N = g_{\mu\nu} dx^\mu dx^\nu + R_{11}^2 (dx^{11} - C_\mu dx^\mu)^2. \]
Let us consider local, purely gravitational interactions in the M theory effective action of the form

$$S = l_{11}^{2k-3} \int d^{11}x \sqrt{-G} D^{2k} R^4.$$ 

Compactify on a circle of (dimensionless) radius $R_{11}$ such that

$$l_{11} = e^{\phi_A/3} l_s, \quad R_{11}^3 = e^{2\phi_A}.$$ 

The length element is given by

$$ds^2 = G_{MN} dx^M dx^N = g_{\mu\nu} dx^\mu dx^\nu + R_{11}^2 (dx^{11} - C_\mu dx^\mu)^2.$$
This leads to a purely gravitational interaction in the type IIA effective action of the form

\[ S = 2\pi l_s^{2k-2} \int d^{10}x \sqrt{-g} e^{2k\phi_\Lambda/3} D^{2k} R^4. \]

Thus the coefficient of the M theory interaction is known if the coefficient of the \( e^{2k\phi_\Lambda/3} \) term is known in the corresponding interaction in the type IIA theory at strong coupling.
This leads to a purely gravitational interaction in the type IIA effective action of the form

\[ S = 2\pi l_s^{2k-2} \int d^{10}x \sqrt{-g} e^{2k\phi_A/3} D^{2k} R^4. \]

Thus the coefficient of the M theory interaction is known if the coefficient of the \( e^{2k\phi_A/3} \) term is known in the corresponding interaction in the type IIA theory at strong coupling.
For \( k \leq 3 \), the IIA interactions are BPS and receive only a finite number of perturbative contributions. Thus the M theory interactions are easily read off from the coefficients of the perturbative amplitudes.

Thus the \( \mathcal{R}^4 \) and \( D^6 \mathcal{R}^4 \) interactions are non-vanishing in the M theory effective action, while the \( D^4 \mathcal{R}^4 \) interaction vanishes.
For $k \leq 3$, the IIA interactions are BPS and receive only a finite number of perturbative contributions. Thus the M theory interactions are easily read off from the coefficients of the perturbative amplitudes.

Thus the $R^4$ and $D^6 R^4$ interactions are non–vanishing in the M theory effective action, while the $D^4 R^4$ interaction vanishes.
In M theory, the coefficient of the $\mathcal{R}^4 \,(D^6\mathcal{R}^4)$ term is fixed by the coefficient of the genus 1 (2) type IIA $\mathcal{R}^4 \,(D^6\mathcal{R}^4)$ amplitude.

The interactions are

$$l^{-3} \zeta(2) \int d^{11}x \sqrt{-G} \mathcal{R}^4$$

and

$$l^3 \zeta(2)^2 \int d^{11}x \sqrt{-GD^6} \mathcal{R}^4$$

dropping overall numerical factors.

The coefficients have a very precise transcendental nature.
In M theory, the coefficient of the $\mathcal{R}^4 \ (D^6 \mathcal{R}^4)$ term is fixed by the coefficient of the genus 1 (2) type IIA $\mathcal{R}^4 \ (D^6 \mathcal{R}^4)$ amplitude.

The interactions are

$$l_{11}^{-3} \zeta(2) \int d^{11}x \sqrt{-G}\mathcal{R}^4$$

and

$$l_{11}^3 \zeta(2)^2 \int d^{11}x \sqrt{-G}D^6\mathcal{R}^4$$

dropping overall numerical factors.

The coefficients have a very precise transcendental nature.
In M theory, the coefficient of the $\mathcal{R}^4 (D^6 \mathcal{R}^4)$ term is fixed by the coefficient of the genus 1 (2) type IIA $\mathcal{R}^4 (D^6 \mathcal{R}^4)$ amplitude.

The interactions are

$$l_{11}^{-3} \zeta(2) \int d^{11}x \sqrt{-G} \mathcal{R}^4$$

and

$$l_{11}^3 \zeta(2)^2 \int d^{11}x \sqrt{-G} D^6 \mathcal{R}^4$$

dropping overall numerical factors.

The coefficients have a very precise transcendental nature.
The interactions $D^{2k} \mathcal{R}^4$ for $k \geq 4$ are non–BPS. Hence the type IIA interactions are expected to receive perturbative contributions from all orders in the genus expansion.

Thus the M theory interactions are difficult to determine because the type IIA coefficients have to extracted at strong coupling.

Can we still make some statements about these non-BPS interactions, for small values of $k$?
The interactions $D^{2k}R^4$ for $k \geq 4$ are non–BPS. Hence the type IIA interactions are expected to receive perturbative contributions from all orders in the genus expansion.

Thus the M theory interactions are difficult to determine because the type IIA coefficients have to extracted at strong coupling.

Can we still make some statements about these non-BPS interactions, for small values of $k$?
The interactions $D^{2k}R^4$ for $k \geq 4$ are non–BPS. Hence the type IIA interactions are expected to receive perturbative contributions from all orders in the genus expansion.

Thus the M theory interactions are difficult to determine because the type IIA coefficients have to extracted at strong coupling.

Can we still make some statements about these non-BPS interactions, for small values of $k$?
For $k = 3n$, the type IIA interaction is of the form

$$S = 2\pi I_6^{n-2} \int d^{10}x \sqrt{-g} e^{2n\phi_A} D^6 n R^4.$$

At weak coupling, this has the structure of the genus $(n + 1)$ string amplitude.
For $k = 3n$, the type IIA interaction is of the form

$$S = 2\pi l_s^{6n-2} \int d^{10}x \sqrt{-g} e^{2n\phi_A} D^{6n} R^4.$$ 

At weak coupling, this has the structure of the genus $(n+1)$ string amplitude.
Though the complete coefficient of the M theory interaction is difficult to determine, it is plausible that for low values of $n$, a part of the coefficient will have features qualitatively described by the genus $(n + 1)$ amplitude, namely the transcendentality.

We shall do the analysis for the $D^{12}R^4$ interaction. The answer we shall get generalizes the transcendental structure for $n = 0$ and $n = 1$. Also there is agreement with a particular supergravity calculation that is valid at strong coupling (to be reviewed later).
Though the complete coefficient of the M theory interaction is difficult to determine, it is plausible that for low values of $n$, a part of the coefficient will have features qualitatively described by the genus $(n + 1)$ amplitude, namely the transcendental structure for $n = 0$ and $n = 1$. Also there is agreement with a particular supergravity calculation that is valid at strong coupling (to be reviewed later).
We shall proceed with this assumption for the $D^{12}\mathcal{R}^4$ interaction.

Thus we want to analyze the genus 3 amplitude for the type IIA $D^{12}\mathcal{R}^4$ interaction. This is the same in the type IIB theory as well.
We shall proceed with this assumption for the $D^{12} \mathcal{R}^4$ interaction.

Thus we want to analyze the genus 3 amplitude for the type IIA $D^{12} \mathcal{R}^4$ interaction. This is the same in the type IIB theory as well.
The spacetime structure of the $R^4$ interaction in flat space has two kinds of contributions:

(i) $t_8 t_8 R^4$
(ii) $\pm \epsilon_{10} \epsilon_{10} R^4$

These follow directly from the string amplitude calculations.

The perturbative contributions to (i) are the same in IIA and IIB, hence this is the part we calculate.
The spacetime structure of the $\mathcal{R}^4$ interaction in flat space has two kinds of contributions:

(i) $t_8 t_8 R^4$

(ii) $\pm \epsilon_1 \epsilon_{10} R^4$

These follow directly from the string amplitude calculations.

The perturbative contributions to (i) are the same in IIA and IIB, hence this is the part we calculate.
Thus we analyze the genus 3 type IIB $D^{12}\mathcal{R}^4$ amplitude.

We shall perform the analysis using the constraints imposed by supersymmetry and S–duality of the type IIB theory.
Thus we analyze the genus 3 type IIB $D^{12}R^4$ amplitude.

We shall perform the analysis using the constraints imposed by supersymmetry and S–duality of the type IIB theory.
In the Einstein frame, the moduli dependent coefficients $f_k(\tau, \bar{\tau})$ of the $D^{2k}R^4$ interactions in the action

$$l_s^{2k-2} \int d^{10}x \sqrt{-g} f_k(\tau, \bar{\tau}) D^{2k}R^4$$

are $SL(2, \mathbb{Z})$ invariant modular forms.

Now $f_k(\tau, \bar{\tau})$ can be constrained using supersymmetry and S–duality.
In the Einstein frame, the moduli dependent coefficients \( f_k(\tau, \bar{\tau}) \) of the \( D^{2k}\mathcal{R}^4 \) interactions in the action

\[
l_s^{2k-2} \int d^{10}x \sqrt{-g} f_k(\tau, \bar{\tau}) D^{2k}\mathcal{R}^4
\]

are \( SL(2, \mathbb{Z}) \) invariant modular forms.

Now \( f_k(\tau, \bar{\tau}) \) can be constrained using supersymmetry and S–duality.
The analysis is done using the Noether procedure.

The action is expanded as

\[ S = S^{(0)} + \sum_{n=3}^{\infty} l_s^{2n} S^{(n)}. \]

The supersymmetry transformation is also expanded as

\[ \delta = \delta^{(0)} + \sum_{n=3}^{\infty} l_s^{2n} \delta^{(n)}. \]
• The analysis is done using the Noether procedure.
• The action is expanded as

\[ S = S^{(0)} + \sum_{n=3}^{\infty} l_s^{2n} S^{(n)}. \]

• The supersymmetry transformation is also expanded as

\[ \delta = \delta^{(0)} + \sum_{n=3}^{\infty} l_s^{2n} \delta^{(n)}. \]
The analysis is done using the Noether procedure.

The action is expanded as

\[ S = S^{(0)} + \sum_{n=3}^{\infty} l_s^{2n} S^{(n)}. \]

The supersymmetry transformation is also expanded as

\[ \delta = \delta^{(0)} + \sum_{n=3}^{\infty} l_s^{2n} \delta^{(n)}. \]
Implementing

\[ \delta S = 0 \]

order by order in the \( l_s \) expansion gives the desired result, on using

\[ \delta^{(0)} S^{(n)} + \delta^{(n)} S^{(0)} + \sum_{p+q=n} \delta^{(p)} S^{(p)} = 0. \]

To actually implement this procedure in a useful way, at a fixed order in the momentum expansion, one looks at the maximally fermionic terms of the form \( \hat{G}^{2k} \lambda^{16} \) and \( \hat{G}^{2k} \psi^* \lambda^{15} \) which should be in the same supermultiplet as the \( D^{2k} \mathcal{R}^4 \) term.
Implementing

$$\delta S = 0$$

order by order in the $l_s$ expansion gives the desired result, on using

$$\delta^{(0)} S^{(n)} + \delta^{(n)} S^{(0)} + \sum_{p+q=n} \delta^{(p)} S^{(p)} = 0.$$ 

To actually implement this procedure in a useful way, at a fixed order in the momentum expansion, one looks at the maximally fermionic terms of the form $\hat{G}^{2k} \lambda^{16}$ and $\hat{G}^{2k} \psi^* \lambda^{15}$ which should be in the same supermultiplet as the $D^{2k} \mathcal{R}^4$ term.
These interactions mix with no other terms in $S^{(k+3)}$ under $\delta^{(0)}$, and one has to find terms in $\delta^{(k+3)}$, as well as terms in $\delta^{(m)}$ and $S^{(n)}$ with $m + n = k + 3$ such that the total supervariation vanishes.

The couplings of these terms in the action as well as the supervariations are $SL(2, \mathbb{Z})$ modular forms of fixed weights, which are further constrained using the closure of the superalgebra.
These interactions mix with no other terms in $S^{(k+3)}$ under $\delta^{(0)}$, and one has to find terms in $\delta^{(k+3)}$, as well as terms in $\delta^{(m)}$ and $S^{(n)}$ with $m + n = k + 3$ such that the total supervariation vanishes.

The couplings of these terms in the action as well as the supervariations are $SL(2, \mathbb{Z})$ modular forms of fixed weights, which are further constrained using the closure of the superalgebra.
These lead to first order differential equations satisfied by the $\hat{G}^{2k}\lambda^{16}$ and $\hat{G}^{2k}\psi^*\lambda^{15}$ couplings, which also holds for other interactions in the same supermultiplet.

These equations are of the form

\[ Df \sim f' + \sum_i g_i h_i, \]

and

\[ \bar{D}f' \sim f + \sum_i k_i l_i. \]
These lead to first order differential equations satisfied by the $\hat{G}^{2k} \lambda^{16}$ and $\hat{G}^{2k} \psi^* \lambda^{15}$ couplings, which also holds for other interactions in the same supermultiplet.

These equations are of the form

$$Df \sim f' + \sum_i g_i h_i,$$

and

$$\bar{D}f' \sim f + \sum_i k_i l_i.$$
Here \( f \) and \( f' \) are coefficients of terms in the same supermultiplet which differ in \( SL(2, \mathbb{Z}) \) weight by 1 unit, and the other coefficients involve terms at lower orders in the \( I_s \) expansion.

Iterating these two equations, we find that the \( D^{2k} \mathcal{R}^4 \) coupling should be expressed as sums of \( SL(2, \mathbb{Z}) \) invariant modular forms, each of which satisfies the Poisson equation

\[
4\tau_2^2 \frac{\partial^2}{\partial \tau \partial \bar{\tau}} f \sim f + \sum_i r_i s_i + \sum_i m_i n_i p_i
\]
on the fundamental domain of moduli space.

Thus, for a fixed value of \( k \) the coupling can be solved recursively once the couplings at lower values of \( k \) are known.
Here $f$ and $f'$ are coefficients of terms in the same supermultiplet which differ in $SL(2, \mathbb{Z})$ weight by 1 unit, and the other coefficients involve terms at lower orders in the $l_s$ expansion.

Iterating these two equations, we find that the $D^{2k} R^4$ coupling should be expressed as sums of $SL(2, \mathbb{Z})$ invariant modular forms, each of which satisfies the Poisson equation

$$4\tau_2^2 \frac{\partial^2}{\partial \tau \partial \bar{\tau}} f \sim f + \sum_i r_i s_i + \sum_i m_i n_i p_i$$

on the fundamental domain of moduli space.

Thus, for a fixed value of $k$ the coupling can be solved recursively once the couplings at lower values of $k$ are known.
Here $f$ and $f'$ are coefficients of terms in the same supermultiplet which differ in $SL(2, \mathbb{Z})$ weight by 1 unit, and the other coefficients involve terms at lower orders in the $l_s$ expansion.

Iterating these two equations, we find that the $D^{2k} \mathcal{R}^4$ coupling should be expressed as sums of $SL(2, \mathbb{Z})$ invariant modular forms, each of which satisfies the Poisson equation

$$4\tau_2^2 \frac{\partial^2}{\partial \tau \partial \bar{\tau}} f \sim f + \sum_i r_i s_i + \sum_i m_i n_i p_i$$

on the fundamental domain of moduli space.

Thus, for a fixed value of $k$ the coupling can be solved recursively once the couplings at lower values of $k$ are known.
From

\[ \delta^{(0)} S^{(3)} + \delta^{(3)} S^{(0)} = 0, \]

we get that the \( \mathcal{R}^4 \) coupling satisfies the Laplace equation

\[ 4 \tau_2^2 \frac{\partial^2}{\partial \tau \partial \bar{\tau}} f^{(0)} = \frac{3}{4} f^{(0)}. \]

Thus

\[ f^{(0)} = E_{3/2}(\tau, \bar{\tau}) = 2 \zeta(3) \tau_2^{3/2} + 4 \zeta(2) \tau_2^{-1/2} + \ldots, \]

leading to the

\[ \int \zeta(2) \int d^{11} x \sqrt{-G} \mathcal{R}^4 \]

term in the M theory effective action.
From

$$\delta^{(0)} S^{(3)} + \delta^{(3)} S^{(0)} = 0,$$

we get that the $\mathcal{R}^4$ coupling satisfies the Laplace equation

$$4\tau_2^2 \frac{\partial^2}{\partial \tau \partial \bar{\tau}} f^{(0)} = \frac{3}{4} f^{(0)}.$$

Thus

$$f^{(0)} = E_{3/2}(\tau, \bar{\tau}) = 2\zeta(3)\tau_2^{3/2} + 4\zeta(2)\tau_2^{-1/2} + \ldots,$$

leading to the

$$l_{11}^{-3} \zeta(2) \int d^{11}x \sqrt{-G}\mathcal{R}^4$$

term in the M theory effective action.
From
\[ \delta^{(0)} S^{(6)} + \delta^{(6)} S^{(0)} + \delta^{(3)} S^{(3)} = 0, \]
we get that the \( D^6 R^4 \) coupling satisfies the Poisson equation
\[ 4\tau_2^2 \frac{\partial^2}{\partial \tau \partial \bar{\tau}} f^{(6)} = 12f^{(6)} - 6E_{3/2}^2. \]
Thus
\[ f^{(6)} = \frac{2}{3} \zeta(3)^2 \tau_2^3 + \frac{4}{3} \zeta(2) \zeta(3) \tau_2 + \frac{8}{5} \zeta(2)^2 \tau_2^{-1} + \frac{32}{945} \zeta(2)^3 \tau_2^{-3} + \ldots, \]
leading to the
\[ l_1^2 \zeta(2)^2 \int d^{11}x \sqrt{-G} D^6 R^4 \]
term in the M theory effective action.
From
\[ \delta^{(0)} S^{(6)} + \delta^{(6)} S^{(0)} + \delta^{(3)} S^{(3)} = 0, \]
we get that the $D^6 R^4$ coupling satisfies the Poisson equation
\[ 4\tau_2^2 \frac{\partial^2}{\partial \tau \partial \bar{\tau}} f^{(6)} = 12f^{(6)} - 6E_{3/2}^2. \]

Thus
\[ f^{(6)} = \frac{2}{3} \zeta(3) \tau_2^3 + \frac{4}{3} \zeta(2) \zeta(3) \tau_2 + \frac{8}{5} \zeta(2)^2 \tau_2^{-1} + \frac{32}{945} \zeta(2)^3 \tau_2^{-3} + \ldots, \]
leading to the
\[ l_{11}^3 \zeta(2)^2 \int d^{11}x \sqrt{-GD^6 R^4} \]
term in the M theory effective action.
Consider the constraints coming from

\[ \delta^{(0)} S^{(9)} + \delta^{(9)} S^{(0)} + \delta^{(3)} S^{(6)} + \delta^{(6)} S^{(3)} + \delta^{(4)} S^{(5)} + \delta^{(5)} S^{(4)} = 0. \]

For every $SL(2, \mathbb{Z})$ invariant modular form in the $D^{12} R^4$ coupling, $\delta^{(3)} S^{(6)} + \delta^{(6)} S^{(3)}$ contributes source terms of the form

\[ \mu E^{3/2} f^{(6)} + \nu E^{3/2} \]

in the Poisson equation.

What about the source terms from $\delta^{(4)} S^{(5)} + \delta^{(5)} S^{(4)}$?

Recall $S^{(4)}$ and $\delta^{(4)}$ vanish on–shell. Hence visible only in some off–shell formalism.
Consider the constraints coming from
\[ \delta^{(0)} S^{(9)} + \delta^{(9)} S^{(0)} + \delta^{(3)} S^{(6)} + \delta^{(6)} S^{(3)} + \delta^{(4)} S^{(5)} + \delta^{(5)} S^{(4)} = 0. \]

For every \( SL(2, \mathbb{Z}) \) invariant modular form in the \( D^{12}R^4 \) coupling, \( \delta^{(3)} S^{(6)} + \delta^{(6)} S^{(3)} \) contributes source terms of the form
\[ \mu E_{3/2} f^{(6)} + \nu E_{3/2}^3 \]
in the Poisson equation.

What about the source terms from \( \delta^{(4)} S^{(5)} + \delta^{(5)} S^{(4)} \)?

Recall \( S^{(4)} \) and \( \delta^{(4)} \) vanish on–shell. Hence visible only in some off–shell formalism.
Consider the constraints coming from
\[ \delta^{(0)} S^{(9)} + \delta^{(9)} S^{(0)} + \delta^{(3)} S^{(6)} + \delta^{(6)} S^{(3)} + \delta^{(4)} S^{(5)} + \delta^{(5)} S^{(4)} = 0. \]

For every \( SL(2, \mathbb{Z}) \) invariant modular form in the \( D^{12} R^4 \) coupling, \( \delta^{(3)} S^{(6)} + \delta^{(6)} S^{(3)} \) contributes source terms of the form
\[ \mu E_{3/2} f^{(6)} + \nu E_{3/2}^3 \]
in the Poisson equation.

What about the source terms from \( \delta^{(4)} S^{(5)} + \delta^{(5)} S^{(4)} \)?

Recall \( S^{(4)} \) and \( \delta^{(4)} \) vanish on–shell. Hence visible only in some off–shell formalism.
Consider the constraints coming from
\[ \delta^{(0)} S^{(9)} + \delta^{(9)} S^{(0)} + \delta^{(3)} S^{(6)} + \delta^{(6)} S^{(3)} + \delta^{(4)} S^{(5)} + \delta^{(5)} S^{(4)} = 0. \]

For every $SL(2, \mathbb{Z})$ invariant modular form in the $D^{12} \mathcal{R}^4$ coupling, $\delta^{(3)} S^{(6)} + \delta^{(6)} S^{(3)}$ contributes source terms of the form
\[ \mu E_{3/2} f^{(6)} + \nu E^3_{3/2} \]
in the Poisson equation.

What about the source terms from $\delta^{(4)} S^{(5)} + \delta^{(5)} S^{(4)}$?

Recall $S^{(4)}$ and $\delta^{(4)}$ vanish on–shell. Hence visible only in some off–shell formalism.
However they provide source terms needed for unitarity.

The genus 1 four graviton amplitude has a non–local contribution of the schematic form

\[ \zeta(2) s \ln(-l_s^2 s) R^4 \]

which in the Einstein frame gives a local interaction of the form

\[ \zeta(2) \ln \tau_2 (s + t + u) R^4 \]

which vanishes on–shell.

Thus off–shell, the \( D^2 R^4 \) interaction has an \( SL(2, \mathbb{Z}) \) invariant coupling

\[ Y(\tau, \bar{\tau}) = \zeta(2) \ln \tau_2 + \ldots \]
However they provide source terms needed for unitarity.

The genus 1 four graviton amplitude has a non–local contribution of the schematic form

$$\zeta(2) \ln(-l_s^2 s) \mathcal{R}^4$$

which in the Einstein frame gives a local interaction of the form

$$\zeta(2) \ln \tau_2 (s + t + u) \mathcal{R}^4$$

which vanishes on–shell.

Thus off–shell, the $D^2 \mathcal{R}^4$ interaction has an $SL(2, \mathbb{Z})$ invariant coupling

$$Y(\tau, \bar{\tau}) = \zeta(2) \ln \tau_2 + \ldots$$
However they provide source terms needed for unitarity.

The genus 1 four graviton amplitude has a non–local contribution of the schematic form

$$\zeta(2) s \ln(-l_s^2 s) \mathcal{R}^4$$

which in the Einstein frame gives a local interaction of the form

$$\zeta(2) \ln\tau_2 (s + t + u) \mathcal{R}^4$$

which vanishes on–shell.

Thus off–shell, the $D^2 \mathcal{R}^4$ interaction has an $SL(2, \mathbb{Z})$ invariant coupling

$$Y(\tau, \bar{\tau}) = \zeta(2) \ln\tau_2 + \ldots$$
Thus $\delta^{(4)} S^{(5)} + \delta^{(5)} S^{(4)}$ contributes source terms

$$YE_{5/2}$$

to the Poisson equation, since $E_{5/2}$ is the $D^4 R^4$ coupling.

Hence the $D^{12} R^4$ coupling $f^{(12)}$ is given by (the structure is the same for either spacetime structure)

$$f^{(12)} = \sum_i f_i^{(12)},$$

where each $f_i^{(12)}$ satisfies the Poisson equation

$$4 \tau^2 \frac{\partial^2}{\partial \tau \partial \bar{\tau}} f_i^{(12)} = \lambda_i f_i^{(12)} - \mu_i E_{3/2} f^{(6)} - \nu_i E_{3/2}^3 - \eta_i YE_{5/2}.$$
Thus $\delta^{(4)} S^{(5)} + \delta^{(5)} S^{(4)}$ contributes source terms

$$YE_{5/2}$$

to the Poisson equation, since $E_{5/2}$ is the $D^4 R^4$ coupling.

Hence the $D^{12} R^4$ coupling $f^{(12)}$ is given by (the structure is the same for either spacetime structure)

$$f^{(12)} = \sum_i f_i^{(12)},$$

where each $f_i^{(12)}$ satisfies the Poisson equation

$$4\tau_2^2 \frac{\partial^2}{\partial \tau \partial \bar{\tau}} f_i^{(12)} = \lambda_i f_i^{(12)} - \mu_i E_{3/2} f^{(6)} - \nu_i E_{3/2}^3 - \eta_i YE_{5/2}.$$
To be consistent with perturbative string amplitudes, we have that

$$\lambda_i = s_i(s_i - 1)$$

where $s_i$ is half-integral.

We can thus solve the equation, whose perturbative part is given by
To be consistent with perturbative string amplitudes, we have that

$$\lambda_i = s_i(s_i - 1)$$

where $s_i$ is half–integral.

We can thus solve the equation, whose perturbative part is given by
\[ f_i^{(12)} \sim c_1 i \tau_i^2 + c_2 i \tau_2^{1-s_i} + \alpha_i \zeta(3)^3 \tau_2^{9/2} + \beta_i \zeta(2) \zeta(3)^2 \tau_2^{5/2} + \gamma_i \zeta(2)^2 \zeta(3) \tau_2^{1/2} + \zeta(2)^3 (\epsilon_i + \sigma_i \zeta(3)) \tau_2^{-3/2} + \omega_i \zeta(2)^4 \tau_2^{-7/2} + \eta_i \zeta(2) \left( 2 \zeta(5) \tau_2^{5/2} + \frac{8}{3} \zeta(4) \tau_2^{-3/2} \right) \ln \tau_2 \]

\[ + \eta_i \zeta(2) \zeta(5) \tau_2^{5/2} + \eta_i \zeta(2)^3 \tau_2^{-3/2} + \ldots \]

It agrees with known calculations in string perturbation theory.
It agrees with known calculations in string perturbation theory.
In the string frame, leads to analytic terms at genus 3 of the form
\[ \zeta(2)^3(\Omega_1 + \Omega_2\zeta(3))e^{4\phi_A}D^{12}\mathcal{R}^4, \]
and at genus 4 of the form
\[ \zeta(2)^4 e^{6\phi_A}D^{12}\mathcal{R}^4. \]

Calculations of the 4 graviton amplitude in regularized maximal supergravity at 1 and 2 loops have been done in the limit of large \( e^{\phi_A} \), and yield
\[ \left( \zeta(2)^3\zeta(3)e^{4\phi_A} + \zeta(2)^4 e^{6\phi_A} + \zeta(2)^6 e^{10\phi_A} \right)D^{12}\mathcal{R}^4. \]
In the string frame, leads to analytic terms at genus 3 of the form
\[ \zeta(2)^3(\Omega_1 + \Omega_2 \zeta(3)) e^{4\phi_A} D^{12} \mathcal{R}^4, \]
and at genus 4 of the form
\[ \zeta(2)^4 e^{6\phi_A} D^{12} \mathcal{R}^4. \]
Calculations of the 4 graviton amplitude in regularized maximal supergravity at 1 and 2 loops have been done in the limit of large \(e^{\phi_A}\), and yield
\[ \left( \zeta(2)^3 \zeta(3) e^{4\phi_A} + \zeta(2)^4 e^{6\phi_A} + \zeta(2)^6 e^{10\phi_A} \right) D^{12} \mathcal{R}^4. \]
Natural to assume that the transcendental structure survives at large coupling, and the M theory coupling is of the form

$$\zeta(2)^3(\Omega_1 + \Omega_2\zeta(3)) I_{11}^9 \int d^{11}x \sqrt{-GD^{12}R^4}.$$ 

We also have to do the analysis for $\lambda_i = 63/4, 15/4, -1/4$. 

Anirban Basu
Natural to assume that the transcendental structure survives at large coupling, and the M theory coupling is of the form

$$\zeta(2)^3(\Omega_1 + \Omega_2\zeta(3))\mathcal{I}^9_{11} \int d^{11}x \sqrt{-G}D^{12}\mathcal{R}^4.$$  

We also have to do the analysis for $\lambda_i = 63/4, 15/4, -1/4.$
In short, for $\lambda_i = 63/4, -1/4$ no change in the genus 3 answer.

For $\lambda_i = 15/4$, we solve

$$4\tau_2^2 \frac{\partial^2}{\partial \tau \partial \bar{\tau}} h = \frac{15}{4} h - \sigma_1 E_{3/2}f^{(6)} - \sigma_2 E_{3/2}^3 - \sigma_3 YE_{5/2}.$$

Apart from the terms for generic $\lambda_i$, we also get (including the $c_1i$ and $c_2i$ parts)
In short, for $\lambda_i = 63/4, -1/4$ no change in the genus 3 answer.

For $\lambda_i = 15/4$, we solve

$$4\tau_2^2 \frac{\partial^2}{\partial \tau \partial \bar{\tau}} h = \frac{15}{4} h - \sigma_1 E_{3/2} f^{(6)} - \sigma_2 E_3^{3/2} - \sigma_3 YE_5^{3/2}.$$ 

Apart from the terms for generic $\lambda_i$, we also get (including the $c_{1i}$ and $c_{2i}$ parts)
In short, for $\lambda_i = 63/4, -1/4$ no change in the genus 3 answer.

For $\lambda_i = 15/4$, we solve

$$4\tau_2^2 \frac{\partial^2}{\partial \tau \partial \bar{\tau}} h = \frac{15}{4} h - \sigma_1 E_{3/2} f^{(6)} - \sigma_2 E_{3/2}^3 - \sigma_3 YE_{5/2}.$$ 

Apart from the terms for generic $\lambda_i$, we also get (including the $c_{1i}$ and $c_{2i}$ parts)
h = c_1 \tau_2^{5/2} + c_3 \tau_2^{-3/2} - 4 \left( \frac{\sigma_1}{3} + 3\sigma_2 \right) \zeta(2) \zeta(3)^2 \tau_2^{5/2} \ln \tau_2

+ 8 \left( \frac{1}{5} \left\{ 1 + \frac{2}{189} \zeta(3) \right\} \sigma_1 + 2\sigma_2 \right) \zeta(2)^3 \tau_2^{-3/2} \ln \tau_2

- \frac{\sigma_3}{4} \zeta(2) \zeta(5) \tau_2^{5/2} (\ln \tau_2)^2 + \frac{\sigma_3}{3} \zeta(2) \zeta(4) \tau_2^{-3/2} \ln \tau_2 + \ldots .

- We can fix the coefficients using string theory data.
- Genus 1 non–analytic piece \sim \zeta(2) \zeta(5) \tau_2^{5/2} \ln \tau_2, hence

\sigma_3 = 0, \sigma_1 = -9\sigma_2.
\[
h = c_1 \tau_2^{5/2} + c_3 \tau_2^{-3/2} - 4 \left( \frac{\sigma_1}{3} + 3 \sigma_2 \right) \zeta(2) \zeta(3)^2 \tau_2^{5/2} \ln \tau_2 \\
+ 8 \left( \frac{1}{5} \left\{ 1 + \frac{2}{189} \zeta(3) \right\} \sigma_1 + 2 \sigma_2 \right) \zeta(2)^3 \tau_2^{-3/2} \ln \tau_2 \\
- \frac{\sigma_3}{4} \zeta(2) \zeta(5) \tau_2^{5/2} (\ln \tau_2)^2 + \frac{\sigma_3}{3} \zeta(2) \zeta(4) \tau_2^{-3/2} \ln \tau_2 + \ldots.
\]

We can fix the coefficients using string theory data.

Genus 1 non–analytic piece \( \sim \zeta(2)\zeta(5)\tau_2^{5/2} \ln \tau_2 \), hence

\[
\sigma_3 = 0, \sigma_1 = -9\sigma_2.
\]
\[ h = c_1 \tau_2^{5/2} + c_3 \tau_2^{-3/2} - 4 \left( \frac{\sigma_1}{3} + 3\sigma_2 \right) \zeta(2) \zeta(3)^2 \tau_2^{5/2} \ln \tau_2 \]
\[ + 8 \left( \frac{1}{5} \left\{ 1 + \frac{2}{189} \zeta(3) \right\} \sigma_1 + 2\sigma_2 \right) \zeta(2)^3 \tau_2^{-3/2} \ln \tau_2 \]
\[- \frac{\sigma_3}{4} \zeta(2) \zeta(5) \tau_2^{5/2} (\ln \tau_2)^2 + \frac{\sigma_3}{3} \zeta(2) \zeta(4) \tau_2^{-3/2} \ln \tau_2 + \ldots . \]

We can fix the coefficients using string theory data.

Genus 1 non–analytic piece \( \sim \zeta(2) \zeta(5) \tau_2^{5/2} \ln \tau_2 \), hence

\[ \sigma_3 = 0, \sigma_1 = -9\sigma_2. \]
Unitarity implies $E_{5/2}Y$ non-analytic coupling, hence a $\zeta(2)^3\tau_2^{-3/2}\ln\tau_2$ contribution, thus $\sigma_1 = 0$.

The final equation is

$$h \sim \zeta(2)E_{5/2} \sim \zeta(2)\zeta(5)\tau_2^{5/2} + \zeta(2)^3\tau_2^{-3/2} + \ldots$$

Hence the genus 3 contribution remains the same.
Unitarity implies $E_{5/2}Y$ non–analytic coupling, hence a $\zeta(2)^3\tau_2^{-3/2}\ln\tau_2$ contribution, thus $\sigma_1 = 0$.

The final equation is

$$h \sim \zeta(2)E_{5/2} \sim \zeta(2)\zeta(5)\tau_2^{5/2} + \zeta(2)^3\tau_2^{-3/2} + \ldots.$$ 

Hence the genus 3 contribution remains the same.
Unitarity implies $E_{5/2} Y$ non–analytic coupling, hence a $\zeta(2)^3 \tau_2^{-3/2} \ln \tau_2$ contribution, thus $\sigma_1 = 0$.

The final equation is

$$h \sim \zeta(2) E_{5/2} \sim \zeta(2) \zeta(5) \tau_2^{5/2} + \zeta(2)^3 \tau_2^{-3/2} + \ldots$$

Hence the genus 3 contribution remains the same.
There could be other contributions to the $D^{12}R^4$ term in M theory. Our calculation and matching with the supergravity analysis suggests that the interaction has at least the terms

$$\zeta(2)^3(\Omega_1 + \Omega_2 \zeta(3)) l_1^9 \int d^{11}x \sqrt{-G} D^{12}R^4.$$
There could be other contributions to the $D^{12}R^4$ term in M theory.

Our calculation and matching with the supergravity analysis suggests that the interaction has at least the terms

$$\zeta(2)^3(\Omega_1 + \Omega_2\zeta(3))l_{11}^9 \int d^{11}x \sqrt{-G}D^{12}R^4.$$
Natural to try to generalize to higher values of $n$.

$D^{18}R^4$ interaction is the next one. The couplings for these coefficients satisfy Poisson equations with non–BPS source terms like the couplings for the $D^8R^4$, $D^{10}R^4$ and $D^{12}R^4$ interactions.

Analogous analysis using known perturbative amplitudes using supersymmetry yields a genus 4 amplitude with coefficient $\sim \zeta(2)^4 (\tilde{\Omega}_1 + \tilde{\Omega}_2 \zeta(3) + \tilde{\Omega}_3 \zeta(5))$.

It is plausible the structure of transcendentality continues at strong coupling showing that the $D^{18}R^4$ M theory interaction has at least the terms

$$\int d^{11} x \sqrt{-GD^{18}R^4}.$$
Natural to try to generalize to higher values of $n$.

$D^{18}R^4$ interaction is the next one. The couplings for these coefficients satisfy Poisson equations with non–BPS source terms like the couplings for the $D^8R^4$, $D^{10}R^4$ and $D^{12}R^4$ interactions.

Analogous analysis using known perturbative amplitudes using supersymmetry yields a genus 4 amplitude with coefficient $\sim \zeta(2)^4(\tilde{\Omega}_1 + \tilde{\Omega}_2\zeta(3) + \tilde{\Omega}_3\zeta(5))$.

It is plausible the structure of transcendentality continues at strong coupling showing that the $D^{18}R^4$ M theory interaction has at least the terms

$$I_{11}^{15} \zeta(2)^4(\tilde{\Omega}_1 + \tilde{\Omega}_2\zeta(3) + \tilde{\Omega}_3\zeta(5)) \int d^{11}x \sqrt{-GD^{18}R^4}.$$
Natural to try to generalize to higher values of $n$.

$D^{18}\mathcal{R}^4$ interaction is the next one. The couplings for these coefficients satisfy Poisson equations with non–BPS source terms like the couplings for the $D^8\mathcal{R}^4$, $D^{10}\mathcal{R}^4$ and $D^{12}\mathcal{R}^4$ interactions.

Analogous analysis using known perturbative amplitudes using supersymmetry yields a genus 4 amplitude with coefficient $\sim \zeta(2)^4(\tilde{\Omega}_1 + \tilde{\Omega}_2\zeta(3) + \tilde{\Omega}_3\zeta(5))$.

It is plausible the structure of transcendentality continues at strong coupling showing that the $D^{18}\mathcal{R}^4$ M theory interaction has at least the terms

$$l_{11}^{15}\zeta(2)^4(\tilde{\Omega}_1 + \tilde{\Omega}_2\zeta(3) + \tilde{\Omega}_3\zeta(5)) \int d^{11}x \sqrt{-G}D^{18}\mathcal{R}^4.$$
Natural to try to generalize to higher values of $n$.

$D^{18}\mathcal{R}^4$ interaction is the next one. The couplings for these coefficients satisfy Poisson equations with non–BPS source terms like the couplings for the $D^8\mathcal{R}^4$, $D^{10}\mathcal{R}^4$ and $D^{12}\mathcal{R}^4$ interactions.

Analogous analysis using known perturbative amplitudes using supersymmetry yields a genus 4 amplitude with coefficient $\sim \zeta(2)^4(\tilde{\Omega}_1 + \tilde{\Omega}_2\zeta(3) + \tilde{\Omega}_3\zeta(5))$.

It is plausible the structure of transcendentality continues at strong coupling showing that the $D^{18}\mathcal{R}^4$ M theory interaction has at least the terms

$$\int_{\mathbb{R}^{11}} \zeta(2)^4(\tilde{\Omega}_1 + \tilde{\Omega}_2\zeta(3) + \tilde{\Omega}_3\zeta(5)) \sqrt{-G} d^{11}x \int d^{11}x \sqrt{-G} D^{18}\mathcal{R}^4.$$
Important to understand better the role of supersymmetry.

Non–BPS interactions in theories with maximal supersymmetry might be tightly constrained.
Important to understand better the role of supersymmetry.

Non–BPS interactions in theories with maximal supersymmetry might be tightly constrained.