Fluctuation bounds for interface free energies of spin glasses

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Spin Glasses and Related Topics
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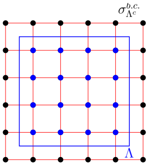
The Edwards-Anderson Model and the RFIM

Consider $\Lambda \subset \mathbb{Z}^d$ a finite box and $E(\Lambda)$ the corresponding edges.

- Ising spin glass or EA model for $\sigma \in \{-1,+1\}^\Lambda$

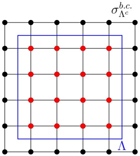
$$H_{\Lambda,\omega}(\sigma) = -\sum_{(x,y)\in E(\Lambda)} \omega_{xy}\sigma_x\sigma_y - \sum_{x\in\Lambda, y\in\Lambda^c} \omega_{xy}\sigma_x\sigma_y^{b.c.}$$



$\omega = (\omega_{xy}; (x,y) \in E(\mathbb{Z}^d))$ i.i.d. with continuous distribution $\mathbb{P}$.

- Random Field Ising Model (RFIM)

$$\widetilde{H}_{\Lambda,\omega}(\sigma) = -\sum_{(x,y)\in E(\Lambda)} J\sigma_x\sigma_y - \sum_{x\in\Lambda, y\in\Lambda^c} J\sigma_x\sigma_y^{b.c.} - \sum_{x\in\Lambda} \omega_x\sigma_x$$



$(\omega_x, x \in \mathbb{Z}^d)$ are i.i.d. random variables.

Variance bounds for the Free Energy

We want bounds on the fluctuations of the free energy:

$$F_\Lambda(\omega) = \log \sum_{\sigma \in \{-1,+1\}^\Lambda} \exp -\beta H_{\Lambda,\omega}(\sigma) \qquad \beta > 0$$

Theorem (Wehr-Aizenman '90, Chatterjee '09)
For the EA and RFIM model on $\mathbb{Z}^d$,

$$Var\ F_\Lambda(\omega) \geq C(\beta)\ |\Lambda|$$

Our goals

1. Study the fluctuations of free energy difference between b.c.
   Interface Free Energy
2. Understand the impact on the structure of the Gibbs states.

Interface Free Energy

Let $\Gamma, \Gamma' \in \mathcal{G}_\omega(\beta)$, two Gibbs states at disorder ω and inv. temp. β.
DLR equations

Interface free energy:

$$F_\Lambda(\omega, \Gamma, \Gamma') = \log \Gamma\Big(\exp \beta H_{\Lambda,\omega}(\sigma_\Lambda, \sigma_{\Lambda^c})\Big) - \log \Gamma'\Big(\exp \beta H_{\Lambda,\omega}(\sigma_\Lambda, \sigma_{\Lambda^c})\Big)$$

- By DLR, this reduces to

$$F_\Lambda(\omega, \Gamma, \Gamma') = \log \frac{\Gamma\Big(Z^{-1}_{\Lambda,\omega}(\beta, \sigma_{\Lambda^c})\Big)}{\Gamma'\Big(Z^{-1}_{\Lambda,\omega}(\beta, \sigma'_{\Lambda^c})\Big)}$$

- At $\beta = \infty$, the analogue is the difference of energies of σ and σ' in Λ ground states in $\mathbb{Z}^d$ at disorder ω.

RFIM: The Aizenman-Wehr theorem

Theorem (Aizenman-Wehr '90)

In $d = 2$, for all $\beta > 0$, there is only one Gibbs state of the RFIM:

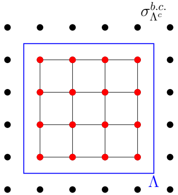
$$\boxed{\#\mathcal{G}_\omega(\beta) = 1 \quad \omega\text{-}a.s.}$$

Rounding of phase transition

- The proof is partially based on the argument by Imry & Ma '85.
- In $d \geq 3$, RFIM exhibits a phase transition (Imbrie '85, Bricmont & Kupiainen '87).

Variance bounds and Gibbs States

In $d = 2$, absence of phase transition in AW is shown by contradiction:



UPPER bound
$|F_\Lambda(\omega, \Gamma, \Gamma')| \leq C|\partial\Lambda|$ ω-a.s. for any Γ, Γ'

The r.v. $|F_\Lambda(\omega, \Gamma, \Gamma')|/|\partial\Lambda|$ is bounded.

Variance LOWER bound

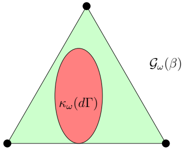
- The r.v. $F_\Lambda/|\Lambda|^{1/2}$ is unbounded
- $$\liminf_{\Lambda \uparrow \mathbb{Z}^d} \mathbb{E}[\exp tF_\Lambda/|\Lambda|^{1/2}] \geq e^{Ct^2} \sim \text{Martingale CLT}$$
- $\boxed{\text{If } \Gamma \neq \Gamma', \text{ Var } F_\Lambda(\omega, \Gamma, \Gamma') \geq C|\Lambda|}$
- Domination by the $(+)$-b.c. and the $(-)$-b.c. states:
$$\Gamma_\omega^-\big(f(\sigma)\big) \leq \Gamma_\omega(f(\sigma)) \leq \Gamma_\omega^+\big(f(\sigma)\big) \qquad \text{FKG inequality}$$
- It suffices to show $\Gamma_\omega^- = \Gamma_\omega^+$ for $\#\mathcal{G}_\omega(\beta) = 1$.

EA: choosing a Gibbs state

For EA

- there is no domination of $(+)$ and $(-)$ states.
- there might be more than one limit state for given b.c.

Way out: sample a state using metastate:



$$\boxed{\kappa_\omega(d\Gamma) \text{ prob. measure on } \mathcal{G}_\omega(\beta)}$$

such that

1. Coupling covariance: If $\omega_B = 0$ except on edges in a box B

$$\kappa_{\omega+\omega_B}(d\Gamma) = \kappa_\omega(d\, L_{\omega_B}\Gamma) \qquad \text{where } L_{\omega_B}\Gamma(\cdots) = \frac{\Gamma(\cdot \exp -\beta H_{B,\omega_B}(\sigma))}{\Gamma(\exp -\beta H_{B,\omega_B}(\sigma))}$$

2. Translation covariance: for any translation T, $\kappa_{T\omega}(d\Gamma) = \kappa_\omega(d\, T\Gamma)$.

The interface free energy $F_\Lambda(\omega, \Gamma, \Gamma')$ is now a r.v. over

$$\boxed{M = d\mathbb{P}(\omega)\ \kappa_\omega(d\Gamma) \times \kappa'_\omega(d\Gamma')}$$

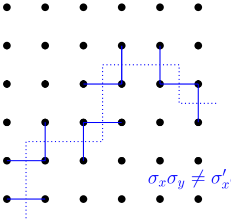
This measure is translation-invariant.

Incongruent states

Some Gibbs states are more physically relevant.
$\Gamma, \Gamma' \in \mathcal{G}_\omega(\beta)$ are incongruent if

$$\liminf_{\Lambda \to \mathbb{Z}^d} \frac{1}{|\Lambda|} \sum_{(x,y)\in E(\Lambda)} 1_{\{|\Gamma(\sigma_x\sigma_y) - \Gamma'(\sigma_x\sigma_y)| > \delta\}} > 0$$

Positive density of edges with different edge-correlation function.



Conjecture (EA model)

- $d = 2$: *At all $\beta > 0$, there is no incongruent Gibbs states. In fact, $\#\mathcal{G}_\omega(\beta) = 1$ a.s.*
- $d > 2$: *Are there incongruent Gibbs states ????*

Our result is to prove a variance LOWER bound for the interface energy in $\mathbb{Z}^d$ under an assumption that incongruent states exist.

Main Result

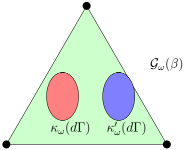
Consider κ_ω and κ'_ω two metastates on $\mathcal{G}_\omega$ and

$M = d\mathbb{P}(\omega)\,\kappa_\omega(d\Gamma)\times\kappa'_\omega(d\Gamma')$ Probability measure on the triplet $(\omega, \Gamma, \Gamma')$.

Assumption (Sufficient for existence of incongruent states)

There exists an edge (x, y) such that with positive $\mathbb{P}$-probability

$$\boxed{\kappa_\omega(\Gamma(\sigma_x\sigma_y)) \neq \kappa'_\omega(\Gamma'(\sigma_x\sigma_y))\ .}$$



Theorem (A-Newman-Stein-Wehr ’14)

Under the above assumption, there exists $C > 0$ such that

$$\boxed{Var_M\big(F_\Lambda(\omega, \Gamma, \Gamma')\big) \geq C|\Lambda|}$$

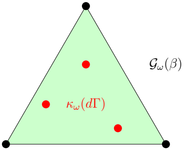
Lower bound for the variance of the interface free energies in $\mathbb{Z}^d$

Corollary in $d = 2$

Corollary (A-Newman-Stein-Wehr '14)

In $d = 2$, a metastate κ_ω cannot be supported on a countable infinite number of incongruent states

$$\kappa_\omega(d\Gamma) = \sum_{i=1}^{\infty} p_\omega^i \ \delta_{\Gamma_\omega^i} \ .$$



Of course, still far from $\#\mathcal{G}_\omega(\beta) = 1$.

Proof.

- Use p_ω^i as tags (like + and − in RFIM): $\{p_\omega^i\}$ is translation-invariant.
- $\kappa_\omega = \delta_{\Gamma_\omega^i}$ and $\kappa_\omega' = \delta_{\Gamma_\omega^{i'}}$ where $p_\omega^i \neq p_\omega^{i'}$ are metastates.
- On one hand: $\mathrm{Var}_M\big(F_\Lambda(J,\Gamma,\Gamma')\big) \geq C|\Lambda|$.
  $\liminf_{\Lambda\uparrow\mathbb{Z}^d} \mathbb{E}[\exp tF_\Lambda/|\Lambda|^{1/2}] \geq e^{Ct^2}$ Martingale CLT
- On the other hand: upper bound

$$\frac{|F_\Lambda(\omega,\Gamma,\Gamma')|}{|\Lambda|^{1/2}} \leq C\frac{1}{|\partial\Lambda|}\sum_{(x,y)\in E(\partial\Lambda)} |J_{xy}| \leq C'$$

□

Possible directions

The interest of the lower bound is to rule out more assumptions

1. $d = 2$ Prove that there is no incongruent states in $\mathbb{Z}^2$.

$$\mathrm{Var}_M(F_\Lambda) = \underbrace{M\Big(\mathrm{Var}_M(F_\Lambda|\omega_\Lambda)\Big)}_{\text{Fluct. of b.c.}} + \underbrace{\mathrm{Var}_M\Big(M(F_\Lambda|\omega_\Lambda)\Big)}_{\text{Fluct. of couplings in }\Lambda}$$

2. $d > 2$ Find a variance UPPER bound to get a contradiction

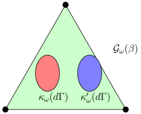
$$\boxed{\mathrm{Var}\left(\log \frac{Z^{per.}_{\Lambda,\omega}(\beta)}{Z^{anti}_{\Lambda,\omega}(\beta)}\right) \leq C|\partial\Lambda|}$$

(Aizenman & Fisher, Newman & Stein, Contucci & Giardina)
Other than gauge-related ?

Picture of the proof

We want to show $\mathrm{Var}_M\big(F_\Lambda(\omega,\Gamma,\Gamma')\big) \geq C|\Lambda|$ for $M = d\mathbb{P}(\omega)\ \kappa_\omega(d\Gamma) \times \kappa'_\omega(d\Gamma')$ under

$$\kappa_\omega(\Gamma(\sigma_x\sigma_y)) \neq \kappa'_\omega(\Gamma'(\sigma_x\sigma_y)) \qquad \text{w.p.p.}$$

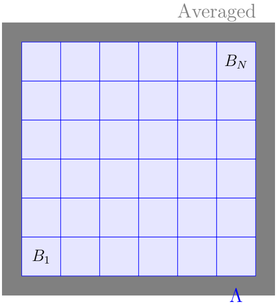


We have

$$\mathrm{Var}_M\left(F_\Lambda\right) \geq \underbrace{\mathrm{Var}_M\Big(M\big(F_\Lambda|\omega_\Lambda\big)\Big)}_{\text{Fluct. of couplings in }\Lambda}$$

Divide Λ into equally sized blocks $B_1, \ldots B_k, \ldots, B_N$ where $N = c|\Lambda|$.

$$\mathrm{Var}_M\left(F_\Lambda\right) \geq \sum_{k=1}^{N} \mathrm{Var}_M\Big(M(F_\Lambda|\omega_{B_k})\Big)$$



To show

1. $\mathrm{Var}_M\Big(M(F_\Lambda|\omega_{B_k})\Big) = \mathrm{Var}_M\Big(M(F_\Lambda|\omega_{B_1})\Big)$
2. $\mathrm{Var}_M\Big(M(F_\Lambda|\omega_{B_1})\Big) > c'$ for $c' > 0$ independent of Λ.

Picture of the proof

$$\text{Var } M(F_\Lambda|\omega_B) = \frac{1}{2}\int d\mathbb{P}(\omega_B)\int d\mathbb{P}(\omega'_B)\,\Big\{M(F_\Lambda|\omega_B) - M(F_\Lambda|\omega'_B)\Big\}^2$$

$$= \frac{1}{2}\int d\mathbb{P}(\omega_B)\int d\mathbb{P}(\omega'_B)\left\{\int_{\omega'_B\to\omega_B} \nabla_B M(F_\Lambda|z_B)\cdot dz_B\right\}^2$$

Lemma

For any $(x,y)\in E(B)$

$$\boxed{\frac{\partial}{\partial\omega_{xy}} M\big(F_\Lambda|\omega_B\big) = \beta\ M\big(\Gamma(\sigma_x\sigma_y) - \Gamma'(\sigma_x\sigma_y)\big|\omega_B\big)\ a.s.}$$

Do not depend on Λ AND Translation invariant

Let ω_{B^c} be the couplings ω where the couplings in B are set to 0:

$$\kappa_\omega(d\Gamma) = \kappa_{\omega_{B^c}}(dL_{\omega_B}\Gamma)\ \text{Coupling covariance}$$

$$L_{\omega_B}\Gamma\Big(\exp\beta H_{\Lambda,\omega}(\sigma)\Big) = \Gamma\Big(\exp\beta H_{\Lambda,\omega}(\sigma)\exp-\beta H_{B,\omega}(\sigma)\Big)/\Gamma\Big(\exp-\beta H_{\omega,B}(\sigma)\Big)$$

Picture of the proof

$$\text{Var } M(F_\Lambda|\omega_B)$$
$$= \frac{\beta^2}{2} \int d\mathbb{P}(\omega_B) d\mathbb{P}(\omega'_B) \Big\{ \int_{\omega'_B \to \omega_B} \underbrace{\sum_{(x,y)\in E(B)} M\big(\Gamma(\sigma_x\sigma_y) - \Gamma'(\sigma_x\sigma_y)\big| z_B\big)\ dz_{xy}}_{\nabla_B M(F_\Lambda|z_B)\cdot dz_B} \Big\}^2$$

The assumption implies that for B large enough

$$M\Big(\Gamma(\sigma_x\sigma_y) - \Gamma'(\sigma_x\sigma_y)|\omega_B\Big) \neq 0 \text{ w.p.p.}$$

since as $B \to \mathbb{Z}^d$

$$M\Big(\Gamma(\sigma_x\sigma_y) - \Gamma'(\sigma_x\sigma_y)\big|\omega_B\Big) \to \kappa_\omega\Big(\Gamma(\sigma_x\sigma_y)\Big) - \kappa'_\omega\Big(\Gamma(\sigma_x\sigma_y)\Big)$$

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Thank you!