

On properties of Parisi measures

Wei-Kuo Chen

Department of Mathematics
University of Chicago

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Mixed p -spin model

- Configuration space: $\Sigma_N = \{-1, +1\}^N$
- Energy:

$$-H_N(\boldsymbol{\sigma}) = \sum_{p \geq 2} \frac{\beta_p}{N^{(p-1)/2}} \sum_{i_1, \dots, i_p=1}^N g_{i_1, \dots, i_p} \sigma_{i_1} \cdots \sigma_{i_p},$$

where g_{i_1, \dots, i_p} are i.i.d. standard Gaussian.

- Gibbs measure: G_N
- Free energy: $F_N = \frac{1}{N} \mathbb{E} \log \sum_{\boldsymbol{\sigma}} \exp(-H_N(\boldsymbol{\sigma}))$.

- Here,

$$\mathbb{E} H_N(\boldsymbol{\sigma}^1) H_N(\boldsymbol{\sigma}^2) = N \xi(R(\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2)),$$

where $R(\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2) = \frac{\boldsymbol{\sigma}^1 \cdot \boldsymbol{\sigma}^2}{N}$ and $\xi(x) = \sum_{p \geq 2} \beta_p^2 x^p$.

Theorem (Parisi '79, Talagrand '06, Panchenko '10)

$$\lim_{N \rightarrow \infty} F_N = \inf_{\mu \in M[0,1]} \mathcal{P}(\mu),$$

where

- $M[0, 1] = \{\text{all prob. meas. on } [0, 1]\}$.
- $\mathcal{P}(\mu) := \Phi_\mu(0, 0) - \frac{1}{2} \int_0^1 q \xi''(q) \mu([0, q]) dq$.
- $\Phi_\mu(q, x)$ is the solution to the Parisi PDE,

$$\partial_q \Phi_\mu(q, x) = -\frac{\xi''(q)}{2} (\partial_{xx} \Phi_\mu(q, x) + \mu([0, q]) (\partial_x \Phi_\mu(q, x))^2)$$

$$\Phi_\mu(1, x) = \log \cosh x.$$

- How many minimizers?
- What is the role played by any such minimizers?

The minimizer of the Parisi functional: Predictions

We call any measure that minimizes the previous variational problem a *Parisi measure*. We denote it by μ_P .

It is expected that

- (a) μ_P is unique.
- (b) The limit law of the overlap under the measure $\mathbb{E}G_N^{\otimes 2}$ is given by μ_P .
- (c) For i.i.d. sampled $(\sigma^\ell)_{\ell \geq 1}$ from G_N , they are ultrametric and the joint law is determined by μ_P .

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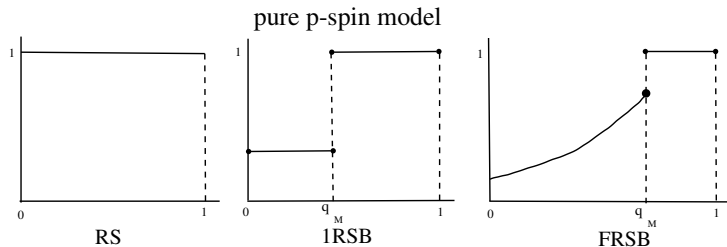
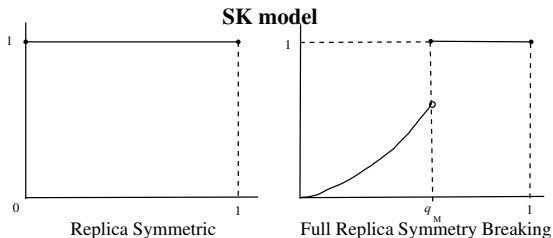
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Quantitative behavior of μ_P : predictions

Let $\alpha(q) = \mu_P([0, q])$.



Physicists' predictions: summary

Any Parisi measure should satisfy the following:

- (1) The origin is contained in the support of the Parisi measure at any temperature.
- (2) One expects FRSB behavior at low temperature.
- (3) Any Parisi measure has a jump discontinuity at q_M at any temperature.

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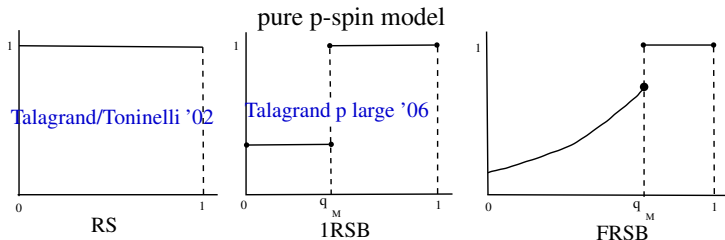
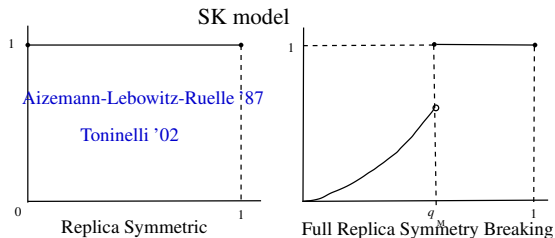
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Our results

Theorem (Uniqueness, Auffinger-Chen '14)

At any temperature $(\beta_p)_{p \geq 2}$, the Parisi measure is unique.

Theorem (Auffinger-Chen '13)

We have that

- $0 \in \text{supp} \mu_P$.
- *If $\beta_2^2 < 1$, then 0 is an isolated point.*

Example (Pure p -spin model with $p \geq 3$)

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Our results: towards FRSB

Theorem (Auffinger-Chen '13)

Suppose that

$$\xi(1) > \frac{1}{3} \sqrt{\xi'(1)} 2^{\frac{\xi'(1)}{\xi(1)} + 5}.$$

Then the Parisi measure is neither RS nor 1RSB.

Example (Pure p -spin model with $p \geq 2$)

Neither RS nor 1RSB if $\beta_p > 2^{p+5} \sqrt{p}/3$.

Theorem (Regularity, Auffinger-Chen '13)

- Suppose that $u_\ell^-, u_\ell^+ \in \text{supp} \mu_P$ for all $\ell \geq 1$ satisfy $u_\ell^- \uparrow u_0$ and $u_\ell^+ \downarrow u_0$. Then μ_P is continuous at u_0 .
- If $(a, b) \subset \text{supp} \mu_P$ for some $0 \leq a < b \leq 1$, then $\alpha(r) := \mu_P([0, r])$ is infinitely differentiable on $[a, b)$.

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Our results: top of the support

Theorem (Auffinger-Chen '13)

Suppose that ξ satisfies $1/\sqrt{2} < \beta_2 \leq 3/2\sqrt{2}$ and

$$\frac{\xi'''(1)}{6} + \frac{2}{3}\sqrt{\xi''(1)} \leq 1.$$

Then the Parisi measure has a jump discontinuity at q_M .

Example (SK model)

If $1/\sqrt{2} < \beta_2 \leq 3/2\sqrt{2}$, then q_M is a jump discontinuity.

Whether there are infinitely many points in the support of μ_P or not is still unclear.

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The Parisi formula

Recall Parisi's formula says

$$\lim_{N \rightarrow \infty} \frac{1}{N} \mathbb{E} \log Z_N = \inf_{\mu \in \mathcal{M}[0,1]} \mathcal{P}(\mu),$$

where

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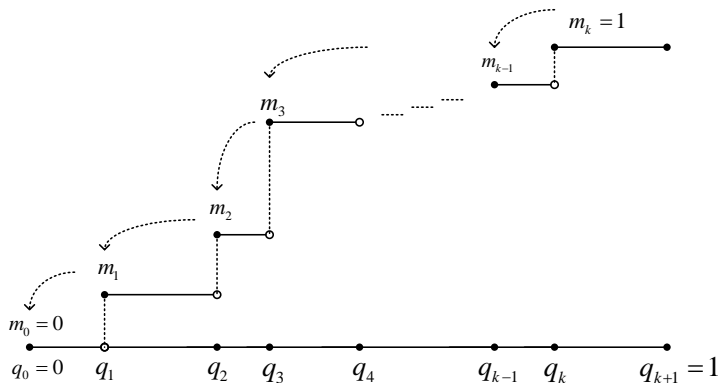
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How to solve the Parisi PDE?

- Consider $\mu \in M([0, 1])$,

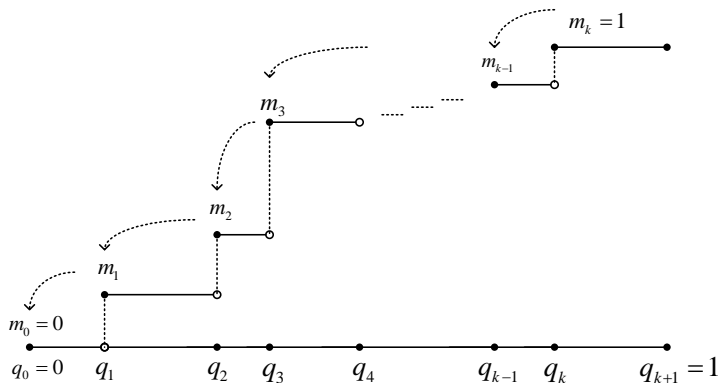


- $\Phi(1, x) = \log \cosh x$.
- Using the Hopf-Cole transformation, for $q \in [q_p, q_{p+1})$,

$$\Phi(q, x) = \frac{1}{m_p} \log \mathbb{E} \exp m_p \Phi(q_{p+1}, x + z \sqrt{\xi'(q_{p+1}) - \xi'(q)}).$$

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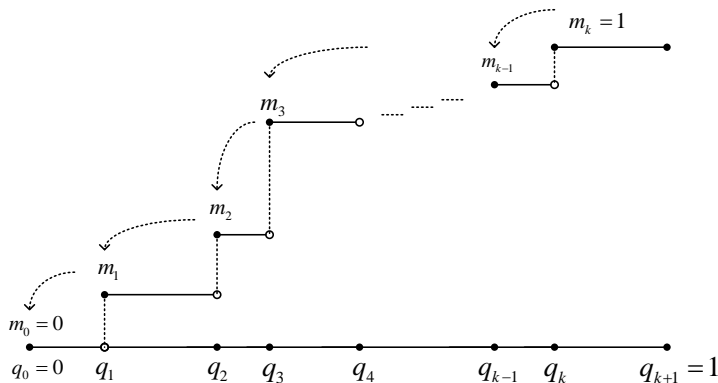


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Know results

- Convexity along one-sided direction if we have stochastic dominance.
 - ▶ Panchenko '05
 - ▶ Chen '13
- Suppose that f is convex. Is the following convex?

$$m \mapsto \frac{1}{m} \log \mathbb{E} \exp mf(Z).$$

- Approaches for Gaussian inequalities:
 - ▶ Ornstein-Uhlenbeck semi-group
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Variational representation for Φ_μ

Let $\mu \in M[0, 1]$. For $0 \leq s \leq t \leq 1$, define

- $D[s, t]$: all prog. meas. $u = (u(w))_{s \leq w \leq t}$ w.r.t. W with $\|u\|_\infty \leq 1$.
- $C_\mu(u, x) = \Phi_\mu \left(t, x + \int_s^t \mu([0, w]) \xi''(w) u(w) dw + \int_s^t \xi''(w)^{1/2} dW(w) \right)$.
- $L_\mu(u) = \frac{1}{2} \int_s^t \mu([0, w]) \xi''(w) u(w)^2 dw$.
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Theorem (Variational representation for the Parisi PDE)

$$\Phi_\mu(s, x) = \max_{u \in D[s, t]} F_\mu(u, x).$$

The maximum is attained by $u_\mu^(w) = \partial_x \Phi_\mu(w, X(w))$, where $(X(w))_{s \leq w \leq t}$ satisfies*

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Proof of convexity of $\mathcal{P}(\mu)$

Let $s = 0$ and $t = 1$.

- $D[0, 1]$: all prog. meas. $u = (u(w))_{0 \leq w \leq 1}$ w.r.t. W with $\|u\|_\infty \leq 1$.
- $C_\mu(u, x) = \log \cosh \left(x + \int_0^1 \mu([0, w]) \xi''(w) u(w) dw + \int_0^1 \xi''(w)^{1/2} dW(w) \right)$.
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Let $\mu_\theta := (1 - \theta)\mu_0 + \theta\mu_1$. Then

$$\begin{aligned} C_{\mu_\theta}(u, x) &\leq (1 - \theta)C_{\mu_0}(u, x) + \theta C_{\mu_1}(u, x), \\ L_{\mu_\theta}(u) &= (1 - \theta)L_{\mu_0}(u) + \theta L_{\mu_1}(u). \end{aligned}$$

So

$$\begin{aligned} F_{\mu_\theta}(u, x) &\leq (1 - \theta)F_{\mu_0}(u, x) + \theta F_{\mu_1}(u, x) \\ \Rightarrow \Phi_{\mu_\theta}(0, x) &\leq (1 - \theta)\Phi_{\mu_0}(0, x) + \theta\Phi_{\mu_1}(0, x). \end{aligned}$$

Proof of convexity of $\mathcal{P}(\mu)$

Let $s = 0$ and $t = 1$.

- $D[0, 1]$: all prog. meas. $u = (u(w))_{0 \leq w \leq 1}$ w.r.t. W with $\|u\|_\infty \leq 1$.
- $C_\mu(u, x) = \log \cosh \left(x + \int_0^1 \mu([0, w]) \xi''(w) u(w) dw + \int_0^1 \xi''(w)^{1/2} dW(w) \right)$.
- $L_\mu(u) = \frac{1}{2} \int_0^1 \mu([0, w]) \xi''(w) u(w)^2 dw$.
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- Assume distinct $\mu_0, \mu_1 \in M[0, 1]$ and $\theta \in (0, 1)$.
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$$\begin{aligned} d\partial_x \Phi_\mu(r, X(r)) &= \xi''(r)^{1/2} \partial_{xx} \Phi_\mu(r, X(r)) dB(r), \quad \forall \mu \in M[0, 1] \\ \Rightarrow \partial_{xx} \Phi_{\mu_0}(r, X_{\mu_0}(r)) &= \partial_{xx} \Phi_{\mu_\theta}(r, X_{\mu_\theta}(r)). \end{aligned}$$

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In the simplest case

Theorem (Boué-Dupuis '98)

Let Z be standard Gaussian and $(W_t)_{0 \leq t \leq 1}$ be a standard BM,

$$\log \mathbb{E} \exp f(Z) = \sup_v \mathbb{E} \left(f \left(W(1) + \int_0^1 v(s) ds \right) - \frac{1}{2} \int_0^1 v(s)^2 ds \right),$$

where the supremum is taken over all progressively measurable v w.r.t. W .

Using change of variables, $v = mu$,

$$\begin{aligned} \frac{1}{m} \log \mathbb{E} \exp mf(Z) &= \sup_v \mathbb{E} \left(f \left(W(1) + \int_0^1 v(s) ds \right) - \frac{1}{2m} \int_0^1 v(s)^2 ds \right) \\ &= \sup_u \mathbb{E} \left(f \left(W(1) + m \int_0^1 u(s) ds \right) - \frac{m}{2} \int_0^1 u(s)^2 ds \right). \end{aligned}$$

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A Gaussian inequality

Theorem (Auffinger-Chen '14)

Let $m_0, m_1 > 0$ and $\theta \in [0, 1]$. If

$$f((1 - \theta)x + \theta y) \leq (1 - \theta)g(x) + \theta h(y),$$

then

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Derivative of the Parisi functional

Consider the mixed p -spin model with external field h . Let $s = 0$ and $t = 1$. Write

$$\mathcal{P}(\mu) = \log 2 + \max_{u \in D[0,1]} \left(F_\mu(u, h) - \frac{1}{2} \int_0^1 \mu([0, w]) w \xi''(w) dw \right).$$

Idea: Define $f(x) = \max_{y \in A} g(x, y)$. Assume that $\exists y(x)$ such that $f(x) = g(x, y(x))$. Then

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Recall $u_\mu^*(w) = \partial_x \Phi_\mu(w, X(w))$, where $(X(w))_{0 \leq w \leq 1}$ satisfies

$$X(w) = x + \int_0^w \alpha(r) \xi''(r) \partial_x \Phi_\mu(r, X(r)) dr + \int_0^w \xi''(r)^{1/2} dB(r).$$

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Let $\mu_0 \in M[0, 1]$. For $\mu \in M[0, 1]$, define $\mu_\theta = (1 - \theta)\mu_0 + \theta\mu$, $\forall \theta \in [0, 1]$.
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$$\left. \frac{d}{d\theta} \mathcal{P}(\mu_\theta) \right|_{\theta=0} = \frac{1}{2} \int_0^1 \xi''(r) (\mu([0, r]) - \mu_0([0, r])) (\mathbb{E}u_{\mu_0}^*(r)^2 - r) dr.$$

TFAE:

- μ_0 is the Parisi measure.
- $\left. \frac{d}{d\theta} \mathcal{P}(\mu_\theta) \right|_{\theta=0} \geq 0$ for all $\mu \in \mathcal{M}$.
- $\left. \frac{d}{d\theta} \mathcal{P}(\mu_\theta) \right|_{\theta=0} \geq 0$ for all $\mu = \delta_q$ with $q \in [0, 1]$.

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First conclusion

Theorem (Chen '14)

For all q in the support of μ_P ,

$$\begin{aligned}\mathbb{E}\partial_x\Phi_{\mu_P}(q, X(q))^2 &= q, \\ \xi''(q)\mathbb{E}\partial_{xx}\Phi_{\mu_P}(q, X(q))^2 &\leq 1,\end{aligned}$$

where for all $s \in [0, 1]$,

$$X(s) = h + \int_0^s \mu_P([0, r])\xi''(r)\partial_x\Phi_{\mu_P}(w, X(w))dw + \int_0^s \xi''(w)^{1/2}dW(w).$$

Assume SK, i.e., $\xi(x) = \beta^2 x^2$. Then

- Toninelli '02: If $\mu_P = \delta_q$ for some $q \in [0, 1]$, then

$$\begin{aligned}\mathbb{E} \tanh^2(\beta z \sqrt{2q} + h) &= q, \\ \beta^2 \mathbb{E} \frac{1}{\cosh^4(\beta z \sqrt{2q} + h)} &\leq 1 \text{ (AT line)}.\end{aligned}$$

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Second conclusion

For $k \geq 0$, set

$$M_k[0, 1] = \{\text{atomic } \mu \in M[0, 1] \text{ with at most } k + 1 \text{ atoms}\}$$

Theorem (Chen '14)

If $\inf_{\mu \in M_{k+1}[0,1]} \mathcal{P}(\mu) = \mathcal{P}(\mu_0)$ for some $\mu_0 \in M_k[0, 1]$, then μ_0 is the Parisi measure.

New result on positivity of the overlap

Positivity holds in the following situations:

- Strong positivity: mixed even p -spin model and $h \neq 0$. (Talagrand)
- Weak positivity: generic mixed p -spin model, $\beta_p \neq 0, \forall p \geq 2$.

Theorem (Chen '14)

Let $q^* = \min \text{supp} \mu_p$. Suppose that ξ is convex on $[-1, 1]$ and

$$\frac{\xi''(s)}{\xi''(s) + \xi''(-s)}$$

is nondecreasing for $s \in (0, 1]$. If ξ is not even, then

$$\mathbb{E} G_N^{\otimes 2}((\sigma^1, \sigma^2) : R(\sigma^1, \sigma^2) < q^* - \varepsilon) \leq K \exp\left(-\frac{N}{K}\right)$$

Note: We do not require $h \neq 0$.

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Positivity holds in different nature

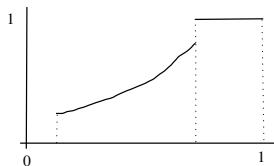


Figure: Positivity of the SK model with $h \neq 0$

Example

$\xi(x) = \beta_{2p}^2 x^{2p} + \beta_{2p+1}^2 x^{2p+1}$ on $[-1, 1]$, where $(2p + 1)\beta_{2p+1}^2 < (2p - 1)\beta_{2p}^2$.

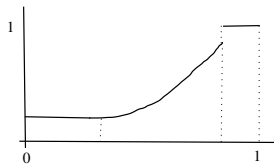


Figure: Positivity of the $(4 + 5)$ -spin model with $h = 0$.