Spin glasses on locally tree like graphs

Amir Dembo, Antoine Gerschenfeld, Andrea Montanari

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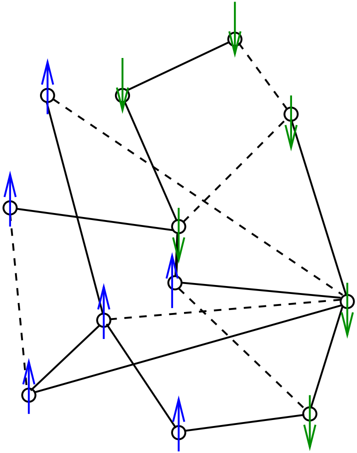
Ising spin glass



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$G_n = (V_n \equiv [n], E_n)$ finite, undirected graphs.

$x_i \in \{+1, -1\}$

$$\mu(\underline{x}) = \frac{1}{Z_{n,\mathbf{J}}(\beta, B)} \exp\left\{\beta \sum_{(ij)\in E_n} J_{ij} x_i x_j + B \sum_{i=1}^{n} x_i\right\}$$

$J_{ij} \in \{+1, -1\}$ uniformly random

[Viana, Bray 1985]

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Locally tree-like graphs (GW case)

$P \equiv \{P_k\}_{k\geq 0}$	Degree distribution, law of L (of mean $\overline{P} > 0$)
$\rho \equiv \{\rho_k\}_{k\geq 0}$	Size-biased P, law of K (degree of uniform edge)
$\mathrm{T}(P, \rho, t)$	t-generations MGW tree (root degree P, else ρ)
$\mathrm{B}_i(t)$	Ball of radius t in G_n centered at node i

Definition

$\{G_n\}$ *converges locally* to $\mathrm{T}(P, \rho, \infty)$ if for uniformly random $I \in [n]$ and fixed t, law of $\mathrm{B}_I(t)$ converges as $n \to \infty$ to $\mathrm{T}(P, \rho, t)$.
$\{G_n\}$ *uniformly sparse* if $\{|\partial I|\}$ is uniformly integrable.

[Benjamini & Schramm 2001, c.f. Aldous & Lyons 2007]

MGW T(P, ρ, t)

P_k

ρ_k

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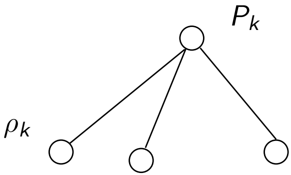

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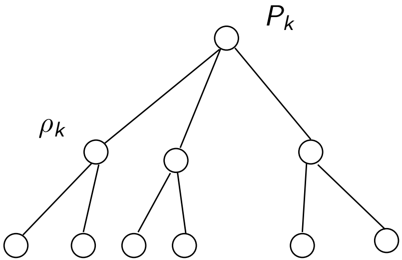

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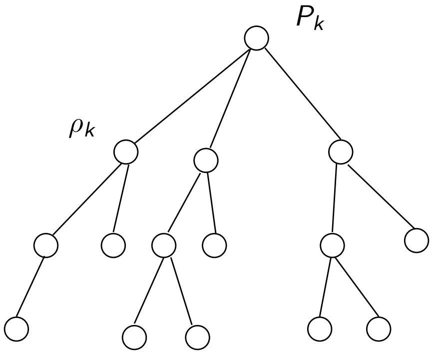

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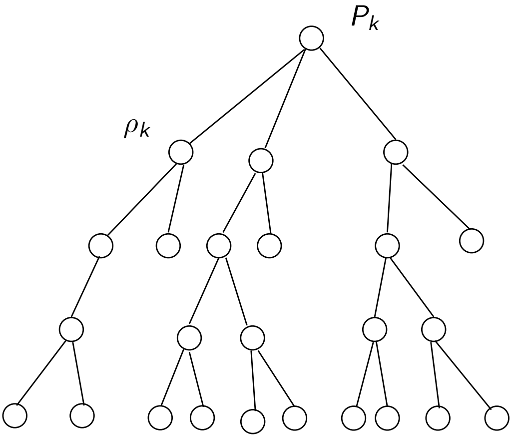

Bethe-Peierls equations

$\mu_{i\to j}(\,\cdot\,) \equiv$ Marginal of x_i when replacing $e^{\beta J_{ij} x_i x_j}$ by 1

$$\mu_{i\to j}(x_i) \approx \frac{1}{z_{i\to j}} e^{Bx_i} \prod_{l\in\partial i\setminus j} \sum_{x_l} e^{\beta J_{il} x_i x_l} \mu_{l\to i}(x_l)$$

Philosophy: approximate local marginals of $\mu(\,\cdot\,)$ in terms of measures on trees.

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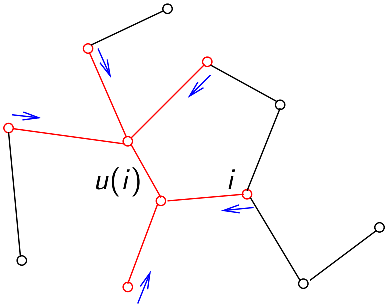
Bethe-Peierls approximation



$F = (U, E_U) \subseteq G$, $\mathrm{diam}(F) \leq t$, such that $\partial i \in U$ or $\partial i \cap U = \{u(i)\}$

$$\mu_U(\underline{x}_U) \approx \nu_U(\underline{x}_U) = \frac{1}{Z_U} \prod_{i \in U \setminus \partial U} e^{Bx_i} \prod_{(i,j) \in E_U} e^{\beta J_{ij} x_i x_j} \prod_{i \in \partial U} \nu_{i \to u(i)}(x_i)\,.$$

$\{\nu_{i \to j}(\,\cdot\,)\}$ $\to$ 'set of messages' (aka cavity fields)

Bethe free energy density

For iid $\theta_i \in \{+\tanh\beta, -\tanh\beta\}$ uniformly at random, independent of K, L and iid $m_i^{(t)}$, let

$$m^{(t+1)} \stackrel{\mathrm{d}}{=} \tanh\left\{B + \sum_{i=1}^{K-1} \operatorname{atanh}(\theta_i\, m_i^{(t)})\right\}, \quad m^{(0)} = 0\,.$$

$$\beta < \beta_c(B, P) \quad \Rightarrow \quad m^{(t)} \stackrel{\mathrm{d}}{\to} m^*$$

m^* fixed point of Bethe or belief propagation recursion (on message sets)

$\beta_c(0, P) = \operatorname{atanh}(1/\sqrt{\rho})$.

For iid m_i^* independent of L, $\gamma = \tanh(B)$,

$$\phi_*(P, \beta, B) = \phi^{\mathrm{v}} - \phi^{\mathrm{e}}$$

$$\phi^{\mathrm{v}} = \log\cosh B + \mathbb{E}\,\log\left\{(1+\gamma)\prod_{i=1}^{L}(1+\theta_i m_i^*) + (1-\gamma)\prod_{i=1}^{L}(1-\theta_i m_i^*)\right\}$$

$$\phi^{\mathrm{e}} = \frac{\overline{P}}{2}\,\mathbb{E}\log(1+\theta_0 m_1^* m_2^*) - \frac{\overline{P}}{2}\,\log\cosh\beta\,.$$

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Free energy density: brief survey

$$\phi = \lim_{n\to\infty} \phi_{n,\mathbf{J}} \qquad \phi_{n,\mathbf{J}} = \frac{1}{n} \log Z_{n,\mathbf{J}}(\beta, B)\,.$$

Theorem

If G_n uniformly sparse, converges locally to $\mathsf{T}(P,\rho,\infty)$ and $\beta < \beta_*(B,P)$, then $\phi = \phi_*(P,\beta,B)$.

Uniformly random G_n, average degree γ

$B = 0$, $\beta_* = \mathrm{atanh}(1/\sqrt{\gamma})$ Guerra, Toninelli 2003
$B \neq 0$, $\beta_* = O(1/\gamma)$ Talagrand 2001, 2003.

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$\beta_*(0,P) = \mathrm{atanh}(1/\sqrt{\overline{\rho}})$; $K \sim k_{\rm typ} \gg 1 \Rightarrow \beta_*(B,P) \simeq \frac{f(B)}{\sqrt{k_{\rm typ}}}$ and $0 < f(B) \uparrow \infty$ with B.

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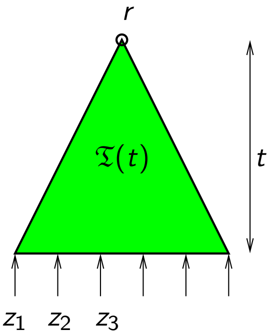
Proof strategy

1. Reduce to expectations of local quantities

$$\frac{d\phi_{n,\mathbf{J}}}{d\beta} = n^{-1} \sum_{(ij)\in E_n} \langle x_i x_j \rangle_{n,\mathbf{J}}$$

   ($\langle \cdot \rangle_{n,\mathbf{J}}$ is expectation under Ising spin glass $\mathbf{J}$ on G_n).
2. Prove convergence of local expectations to tree values.

Convergence to tree values



$\mathfrak{T}$	infinite tree with max degree $k_{\max}$
$\mathfrak{T}(t)$	first t generations
$\mu_{\mathbf{J}}^{t,z}(\,\cdot\,)$	Ising spin glass on $\mathfrak{T}(t)$ boundary condition z
$\mu_{\mathbf{J},r}^{t,z}$	root spin expectation

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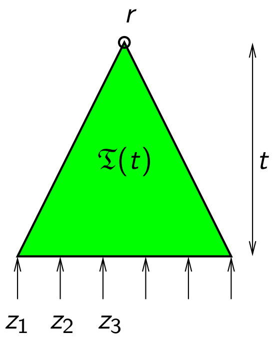

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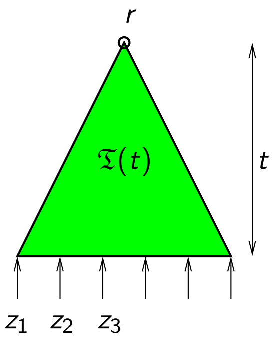

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Uniform (Gibbs measure uniqueness)

$$\mathbb{E}_{\mathsf{T}}\mathbb{E}_{\mathbf{J}} \sup_{z(1),z(2)} |\mu_{\mathbf{J},r}^{t,z(1)} - \mu_{\mathbf{J},r}^{t,z(2)}| \to 0$$

$$\beta = O(1/k_{\text{typ}})$$

Decorrelation on random tree at high temperature

Uniform (Gibbs measure uniqueness)

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Decorrelation at high temperature

Non-uniform

$$\mathbb{E}_{\mathsf{T}} \sup_z \mathbb{E}_{\mathbf{J}} |\mu^{t,z}_{\mathbf{J},r} - \mu^{t,\mathrm{free}}_{\mathbf{J},r}| \le \eta(\beta, B)^t \to 0$$

$$\eta(\beta, B) < 1 \text{ at } \beta \simeq \frac{f(B)}{\sqrt{k_{\mathrm{typ}}}}$$

Decorrelation at high temperature

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