Joint work with A. Barra, P. Contucci and D. Tantari

Multi-species SK model
Guerra's bound

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Outline

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- Related results

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- *Covariance matrix*: $\mathbb{E} H_N(\sigma^1) H_N(\sigma^2) = N\, q_{12}^2$
  where σ^1, σ^2 are two spin configurations and

$$q_{12} = \frac{1}{N} \sum_{i \in I} \sigma_i^1 \sigma_i^2$$

  is called *Overlap*.

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- for each $s \in \mathcal{S}$ the relative densities $\alpha_s = \frac{|I_s|}{N} \in (0,1)$ are independent of N,
- the random coupling J are independent gaussian but **not** identical distributed r.v.,
  $\mathbb{E}J_{ij}^2 = \Delta_{st}$ if $i \in I_s$, $j \in I_t$ for $s, t \in \mathcal{S}$.

Then the covariance matrix becomes

$$\mathbb{E}H_N(\sigma^1)H_N(\sigma^2) = N \sum_{s,t\in\mathcal{S}} \Delta_{st}\alpha_s\alpha_t q_{12}^{(s)} q_{12}^{(t)}$$

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where $\mathbf{q}_{12} \in [-1, 1]^{|\mathcal{S}|}$ is called *Vector Overlap* and defined by

$$\mathbf{q}_{12} = \left(\alpha_s q_{12}^{(s)} \right)_{s \in \mathcal{S}}$$

Guerra's bound

Theorem

If the matrix Δ *is positive semi-definite then the thermodynamical limit of the pressure* $p_N = \frac{1}{N}\mathbb{E}\log\sum_\sigma e^{-H_N(\sigma)}$ *of the M-SK model exist and satisfies the Guerra's bound*

$$\lim_{N\to\infty} p_N = \sup_N p_N \leq \inf_x \mathcal{P}(x)$$

where $\mathcal{P}(x)$ *is a Parisi-like functional and* $x : [0,1]^{|\mathcal{S}|} \to [0,1]$ *is a function with suitable properties and play the role of order parameter of the model.*

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One may think $(m, \mathbf{q}_l)$ as a probability measure on $[0,1]^{\mathcal{S}}$ consisting of atoms at $(q_l^{(1)}, \ldots, q_l^{(s)})$ with jump size $m_{l+1} - m_l$.

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$$x(\mathbf{u}) := \sum_{l=0}^{K} (m_{l+1} - m_l) \prod_{s \in \mathcal{S}} \theta(u^{(s)} - q_l^{(s)})$$

for each $\mathbf{u} = (u^{(s)})_{s \in \mathcal{S}}$ in $[0,1]^{|\mathcal{S}|}$.

The Parisi functional

Given the order parameter x, *i.e.* m and $\mathbf{q}_l$, defined above, for each $s \in \mathcal{S}$, let us define

$$Q_l = \left(\mathbf{q}_l, \Delta\mathbf{q}_l\right)$$

$$Q_l^{(s)} = 2\sum_{t\in\mathcal{S}} \Delta_{st}\alpha_t q_l^{(s)} = 2(\Delta\mathbf{q}_l)_s$$

The Parisi functional is

$$\mathcal{P}(x) = \log 2 + \sum_{s\in\mathcal{S}} \alpha_s f^{(s)} - \frac{1}{2}\sum_{l=1}^{K} m_l\left(Q_l - Q_{l-1}\right)$$

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for each $s \in \mathcal{S}$, $f^{(s)} \equiv f_0^{(s)}$ can be defined by recursively

$$f_K^{(s)} = \log\cosh \sum_{l=1}^{K} \eta_l \left(Q_l^{(s)} - Q_{l-1}^{(s)} \right)^{\frac{1}{2}}$$

$$f_{l-1}^{(s)} = \frac{1}{m_l} \log \mathbb{E}_l e^{m_l f_l^{(s)}}$$

with $(\eta_l)_{l=1,\dots,K}$ *i.i.d.* standard gaussian.

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Consider the probability measures $x^{(s)}$ on $[0, Q_K(s)]$ that satisfy

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$$\mathcal{P}(x) = \log 2 + \sum_{s \in \mathcal{S}} \alpha_s f^{(s)}(0,0) - \frac{1}{2} \int_{\widetilde{\Gamma}} \; x(\mathbf{u}) \;\; \nabla_{\mathbf{u}} \Big(\mathbf{u}, \Delta \mathbf{u} \Big) \cdot d\mathbf{u}$$

$$\frac{\partial f^{(s)}}{\partial x} + \frac{1}{2}\frac{\partial^2 f^{(s)}}{\partial y^2} + \frac{1}{2}x^{(s)}\left(\frac{\partial f^{(s)}}{\partial y}\right)^2 = 0$$

$$f^{(s)}(Q_K^{(s)}, y) = \log\cosh(y).$$

The integral is on a path $\widetilde{\Gamma}$, starting from $\mathbf{0}$ and ending in $\mathbf{1}$, such that $\Gamma \subset \widetilde{\Gamma}$.

The RS approximation

$$p_{RS} = \log 2 + \sum_{s\in\mathcal{S}} \alpha_s f_{RS}^{(s)} + \frac{1}{2}\Big((\mathbf{1}-\mathbf{q}), \Delta(\mathbf{1}-\mathbf{q})\Big),$$

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The optimization on $\mathbf{q}$, gives a system of coupled self consistent equations

$$\sum_{s \in \mathcal{S}} \Delta_{ts} \left[\mathbb{E} \tanh^2 \left(\sqrt{Q^{(s)}} z + h^{(s)} \right) - q^{(s)} \right] = 0,$$

for each $t \in \mathcal{S}$.

Sketch of the proof

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- Easy to obtain super-additivity of the pressure by Guerra-Toninelli argument.
- Upper bound by Guerra's scheme.

Related results

- Proof of the lower bound, (Panchenko '13)

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  Ingredients: ASS scheme $+$ synchronization property.

THANK YOU!