About Eigenvectors for Random Matrices

Spin Glasses and Related Topics (14w5082)

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Outline/Idea/Disclaimer

For mean field spin glasses, an important statistic is the spin-spin overlap.

Random matrix theory is supposedly easier than spin glass theory.

What is the distribution of inner-products of left- and right-eigenvectors of non-Hermitian random matrices?

Complex Ginibre ensemble: $A = (a_{jk})_{j,k=1}^{n}$, $a_{jk} = (X_{jk} + iY_{jk})/\sqrt{2n}$ all IID $\mathcal{N}(0,1)$.

- B. Mehlig and J. T. Chalker, "Statistical properties of eigenvectors in non-Hermitian Gaussian random matrix ensembles," *J. Math.Phys.* (2000).

(1) Motivate the problem, especially using moments.
(2) Describe what is known by Chalker and Mehlig.
(3) Describe what should still be done (and what we can do).

1. Moments for the complex Ginibre ensemble

One commonality to spin glass theory and random matrix theory is the importance of two mathematical tools:

- self-averaging-ness also known as "concentration of measure"
- Gaussian integration by parts
- nonlinear recurrence relations

In spin glasses, these tools lead to the Ghirlanda-Guerra identities. C.f.,

- P. Contucci and C. Giardinà: The Ghirlanda-Guerra identities. *J. Stat. Phys.* **126**(4), 917931 (2007).

In random matrix theory, these tools lead to formulas for the moments of random matrices. See for example Section 2.1 of

- "An Introduction to Random Matrices," G. W. Anderson, A. Guionnet and O. Zeitouni, Cambridge University Press, 2009.

Because of Panchenko's work, we know that the *Extended Ghirlanda Guerra Identities* are enough to generically deduce ultrametricity and Parisi's ansatz:

- Dmitry Panchenko: The Parisi Ultrametricity Conjecture. *Ann. Math.* **177** 383–393 (2013).

In the Gaussian Orthogonal Ensemble, putting the moments together to get the Stieltjes transform allows one to deduce the Wigner semi-circle law.

Again, see for instance

- "An Introduction to Random Matrices," G. W. Anderson, A. Guionnet and O. Zeitouni, Cambridge University Press, 2009.

For the complex Ginibre ensemble $A^{(n)} \in M_n(\mathbb{C})$ is a random matrix

$$(A^{(n)})_{jk} = a_{jk}^{(n)} = \frac{X_{jk} + iY_{jk}}{\sqrt{2n}},$$

for $(X_{jk})_{j,k=1}^{\infty}$ and $(Y_{jk})_{j,k=1}^{\infty}$ IID $\mathcal{N}(0,1)$ random variables.

The "mixed matrix moments" are, for given $r \in \mathbb{N}$ and $p(1), q(1), \dots, p(r), q(r) \in \mathbb{N}$,

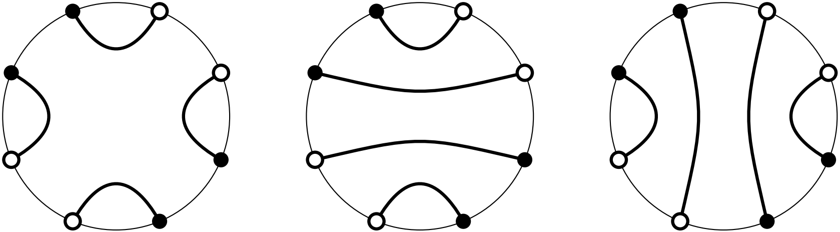
$$M_n(p(1), q(1), \dots, p(r), q(r)) = \frac{1}{n} \operatorname{tr}\Big[(A^{(n)})^{p(1)}((A^{(n)})^*)^{q(1)} \cdots (A^{(n)})^{p(r)}((A^{(n)})^*)^{q(r)}\Big]$$

Then

$$M_n(p(1), q(1), \dots, p(r), q(r)) \stackrel{n\to\infty}{\Longrightarrow} m(p(1), q(1), \dots, p(r), q(r)),$$

$$\# \{ \text{ non-crossing pairings of } (0^{p(1)}, 1^{q(1)}, \dots, 0^{p(r)}, 1^{q(r)}) \ : \ \text{each edge connects one 0 and one 1 } \}.$$

E.g., $m(2,2,2,2) = 3$:



See for instance

- T. Kemp, K. Mahlburg, A. Rattan, C. Smyth: Enumeration of non-crossing pairings on bit strings, *J. Combinat. Theor. A* **118** (2001).

The proof of the limit is an exercise.

Another example: $m(p, p) = 1$ for all p.

2. Chalker and Mehlig's results

Note: $M_n(p(1), q(1), \dots, p(r), q(r))$ depends on the eigenvectors.

Write

$$A^{(n)} = \sum_{\alpha=1}^{n} \lambda_\alpha^{(n)} v_\alpha^{(n)} (w_\alpha^{(n)})^*$$

where $(w_\alpha^{(n)})^* v_\beta^{(n)} = \delta_{\alpha,\beta}$. Then

$$M_n(p(1), q(1), \dots, p(r), q(r)) = \frac{1}{n} \sum_{\substack{\alpha(1),\dots,\alpha(r) \\ \beta(1),\dots,\beta(r)}} (\lambda_{\alpha(1)}^{(n)})^{p(1)} [(w_{\alpha(1)}^{(n)})^* w_{\beta(1)}^{(n)}]$$
$$(\overline{\lambda}_{\beta(1)}^{(n)})^{q(1)} [(v_{\beta(1)}^{(n)})^* v_{\alpha(2)}^{(n)}] (\lambda_{\alpha(2)}^{(n)})^{p(2)} [(w_{\alpha(2)}^{(n)})^* w_{\beta(2)}^{(n)}]$$
$$\cdots (\overline{\lambda}_{\beta(r)}^{(n)})^{q(r)} [(v_{\beta(r)}^{(n)})^* v_{\alpha(1)}^{(n)}]$$

So you need the joint distribution of the eigenvalues and the eigenvector inner-products, what Chalker and Mehlig calculated.

Chalker and Mehlig define $O_n(z)$ such that

$$\frac{1}{n}\,\mathbb{E}\Big[\sum_{\alpha=1}^{n}\|v_\alpha^{(n)}\|^2\|w_\alpha^{(n)}\|^2 f(\lambda_\alpha^{(n)})\Big] = \int_{\mathbb{C}} f(z)O_n(z)\,d^2z$$

and they define $O_n(z_1,z_2)$ such that

$$\frac{1}{n}\,\mathbb{E}\Big[\sum_{\alpha=1}^{n}\sum_{\beta\neq\alpha}[(v_\beta^{(n)})^* v_\alpha^{(n)}][(w_\alpha^{(n)})^* w_\beta^{(n)}]g(\lambda_\alpha^{(n)},\lambda_\beta^{(n)})\Big]$$
$$= \int_{\mathbb{C}}\int_{\mathbb{C}} g(z_1,z_2)\,O_n(z_1,z_2)\,d^2z_1\,d^2z_2\,.$$

Then

$$\frac{1}{n}\,\mathbb{E}[(A^{(n)})^p((A^{(n)})^*)^p] = \int_{\mathbb{C}} |z|^{2p}\,O_n(z)\,d^2z$$
$$+\int_{\mathbb{C}}\int_{\mathbb{C}} z_1^p\bar{z}_2^p O_n(z_1,z_2)\,d^2z_1\,d^2z_2\,.$$

Chalker and Mehlig prove that

$$O_n(z) \sim \frac{n}{\pi}(1-|z|^2)\,, \quad \text{for } |z|<1,$$

and $O_n(z) \to 0$ for $|z| > 1$.

They *argue* that

$$O_n(z_1,z_2) \sim -n^2 \frac{1-|z_+|^2}{\pi^2|\omega|^4}(1-(1+|\omega|^2)e^{-|\omega|^2})\,,$$

for $|z_1|<1$, $|z_2|<1$ where $z_+ = (z_1+z_2)/2$ and $\omega = (z_2-z_1)n^{1/2}$.

The leading order divergences actually do cancel when one uses these formulas to try to verify $m(p,p) = 1$.

But one needs more precise asymptotics to actually verify the formula.

3. More precise asymptotics (and what is needed)

Chalker and Mehlig actually give the exact formula:

$$O_n(z) = \pi^{-1} \exp(-N|z|^2) \sum_{k=0}^{n-1} \frac{(N|z|^2)^k}{k!} (n-k)\,.$$

The reason they can get such a formula is that they can relate it to an integral over the eigenvalue marginal. See their paper and also see Appendix 33 of

- "Random Matrices, 3rd Ed.," Madan L. Mehta, Elsevier, 2004.

Using this it is easy to see that

$$O_n(1 + un^{-1/2}) \sim \frac{n^{1/2}}{\pi\sqrt{2\pi}} e^{-2u^2} - \frac{n^{1/2}}{\pi} (2u)\Phi(-2u)\,.$$

For $O_n(z_1, z_2)$ they can only calculate it exactly when z_1 (or z_2) equals 0.

Otherwise they get a formula involving the determinant of a 5-diagonal matrix.

However, their argument for the precise asymptotic formula is very convincing *in the bulk*.

If we assume their formula in the bulk, then $m(p, p) = 1$ forces the edge scaling of $O_n(z_1, z_2)$ to have two different types of singularities (lower order than the leading-order n divergence that cancelled): a $\log(n)$ divergence and a $n^{3/4}$ divergence.

Guess: This suggests that $O_n(1, 1 - n^{-1/2}r)$ decays like $1/r$ for $r > 0$, but that along the tangential direction of the circle there is a $n^{-1/4}$ correlation length.

How we propose to proceed

One should not need to solve the determinants exactly.

The matrix entries on each diagonal are slowly varying.

So we should be able to use adiabatic theory for slowly varying transfer matrices (or recurrence relations with slowly varying coefficients) to calculate the leading order asymptotics.

This method can potentially work for the 5-diagonal determinant without having an exact (combinatorial) formula.

So far, using this method, we have been able to verify that for the intensity, it is order 1 when $0 < |z| < 1$.

$$d(z)=N\sigma^2(\pi N!\sigma^2)^{-1}\exp\left(-\frac{|z|^2}{\sigma^2}\right)\det\begin{bmatrix} d_{00} & d_{01} & & & 0 \\ d_{10} & & \ddots & & \\ & \ddots & & & \\ & & & & d_{N-3N-2} \\ 0 & & & d_{N-2N-3} & d_{N-2N-2} \end{bmatrix} \tag{24}$$

with $d_{ij}=(\pi j!\sigma^{2j+4})^{-1}\int d^2\lambda\,\overline{\lambda}^i\lambda^j|z-\lambda|^2\exp(-\sigma^{-2}|\lambda|^2)$. Denoting the $(N-1)\times(N-1)$ determinant in Eq. (24) by D_{N-1}, we derive the recursion relation

$$D_{k+1}=(\sigma^{-2}|z|^2+k+1)D_k-\sigma^{-2}|z|^2kD_{k-1}. \tag{25}$$

Using $D_1=1+\sigma^{-2}|z|^2$ and $D_2=2+2\sigma^{-2}|z|^2+\sigma^{-4}|z|^4$, we thus obtain

$$d(z)=\pi^{-1}\exp\left(-\frac{|z|^2}{\sigma^2}\right)\sum_{l=0}^{N-1}\frac{|z|^{2l}}{l!\sigma^{2l}} \tag{26}$$

which corresponds to Eq. (51.1.32) in Ref. 1. In the limit of large N, with $\sigma^2=N^{-1}$, the density of states is

$$d(z)=\begin{cases}\pi^{-1} & \text{for } |z|<1,\\ 0 & \text{otherwise.}\end{cases} \tag{27}$$

Summary

1. There are many analogies between spin glasses and random matrix theory.

2. For spin glasses the Ghirlanda-Guerra identities are key. For random matrix theory nonlinear recurrence relations for the moments are key.

3. For non-Hermitian matrix ensembles, such as the complex Ginibre ensemble, the mixed matrix moments involve both eigenvalues and eigenvector inner-products. This is reminiscent of the spin-spin overlap in SG theory.

4. For the most solvable model, the complex Ginibre ensemble, Chalker and Mehlig did show how to reduce the eigenvector inner-product kernels to (complicated) integrals over the eigenvalue marginals. Therefore, there is hope to calculate exactly.

5. But even to recover the simplest moments from their method $m(p, p)$ one needs to do some more edge asymptotic calculations (and verify their conjectured) formula in the bulk. We are trying to work on that, now.

Thanks for your attention!