Extremal Processes of Gaussian Processes Indexed by Trees

Anton Bovier
with Louis-Pierre Arguin, Nicola Kistler
and Lisa Hartung

Institute for Applied Mathematics Bonn

BIRS, 07.21.2014
Motivation

Spin glasses: What is the structure of ground states for (mean field) spin glasses?

Extreme value theory: What are the extreme values and the extremal process of dependent random processes?

Spatial branching processes: Describe the cloud of spatial branching processes, in particular near their propagation front!

Reaction diffusion equations: Characterise convergence to travelling wave solutions in certain non-linear odes!

This is too hard in general, but we will look at a setting where these questions have a chance to be answered. Branching Brownian motion is at the heart of this setting.
Motivation

- **Spin glasses:** What is the structure of ground states for (mean field) spin glasses?
Motivation

- **Spin glasses:** What is the structure of ground states for (mean field) spin glasses?

- **Extreme value theory:** What are the extreme values and the extremal process of dependent random processes?
Motivation

- **Spin glasses:** What is the structure of ground states for (mean field) spin glasses?

- **Extreme value theory:** What are the extreme values and the extremal process of dependent random processes?

- **Spatial branching processes:** Describe the cloud of spatial branching processes, in particular near their propagation front!
Motivation

- **Spin glasses**: What is the structure of *ground states* for (mean field) spin glasses?

- **Extreme value theory**: What are the extreme values and the extremal process of dependent random processes?

- **Spatial branching processes**: Describe the cloud of spatial branching processes, in particular near their *propagation front*!

- **Reaction diffusion equations**: Characterise convergence to travelling wave solutions in certain non-linear odes!
Motivation

- **Spin glasses:** What is the structure of ground states for (mean field) spin glasses?

- **Extreme value theory:** What are the extreme values and the extremal process of dependent random processes?

- **Spatial branching processes:** Describe the cloud of spatial branching processes, in particular near their propagation front!

- **Reaction diffusion equations:** Characterise convergence to travelling wave solutions in certain non-linear odes!

This is too hard in general, but we will look at a setting where these questions have a chance to be answered. Branching Brownian motion is at the heart of this setting.
Gaussian processes labelled by trees
Gaussian processes on trees

Gaussian processes labelled by trees

- A time-homogeneous tree. Label individuals at time $t$ as $i_1(t), \ldots, i_n(t)$.
Gaussian processes labelled by trees

- A time-homogeneous tree. Label individuals at time $t$ as $i_1(t), \ldots, i_n(t)$.

- Canonical tree-distance: $d(i_\ell(t), i_k(t)) \equiv$ time of most recent common ancestor of $i_\ell(t)$ and $i_k(t)$.
Gaussian processes labelled by trees

- A time-homogeneous tree. Label individuals at time $t$ as $i_1(t), \ldots, i_n(t)$. 
- Canonical tree-distance: $d(i_\ell(t), i_k(t)) \equiv$ time of most recent common ancestor of $i_\ell(t)$ and $i_k(t)$.
- For fixed time horizon $t$, define Gaussian process with covariance
  
  $$\mathbb{E} x_k(r)x_\ell(s) = tA(t^{-1}d(i_k(r), i_\ell(s)))$$

  for $A : [0, 1] \rightarrow [0, 1]$, increasing.
A time-homogeneous tree. Label individuals at time $t$ as $i_1(t), \ldots, i_n(t)(t)$.

Canonical tree-distance: $d(i_\ell(t), i_k(t)) \equiv$ time of most recent common ancestor of $i_\ell(t)$ and $i_k(t)$

For fixed time horizon $t$, define Gaussian process with covariance

$$\mathbb{E} x_k(r)x_\ell(s) = tA(t^{-1}d(i_k(r), i_\ell(s)))$$

for $A : [0, 1] \to [0, 1]$, increasing.

Processes can be constructed as time change of branching Brownian motion branching at the vertices of the trees.
Examples

- **Binary tree, branching at integer times**: \( A(x) = x \) - Branching random walk
- **Step function**: Derrida's Generalised Random Energy Models (GREM)
  - Special case: \( A(x) = 0 \) for \( x < 1 \), \( A(1) = 1 \) - Random energy model (REM), i.e.
  - \( n \) independent random variables \( N(0, t) \)
- **Supercritical Galton-Watson tree**: \( A(x) = x \) - Branching Brownian motion (BBM)
  - General variable speed BBM [Derrida-Spohn, Fang-Zeitouni]
Examples

Binary tree, branching at integer times
Examples

Binary tree, branching at integer times

- $A(x) = x$: Branching random walk
Examples

Binary tree, branching at integer times

- $A(x) = x$: Branching random walk
- A step function: Derrida’s Generalised Random Energy models (GREM)
Examples

Binary tree, branching at integer times

- $A(x) = x$: Branching random walk
- $A$ step function: Derrida’s Generalised Random Energy models (GREM)
- Special case $A(x) = 0, x < 1, A(1) = 1$: Random energy model (REM), i.e. $n(t) \text{iid } \mathcal{N}(0, t)$ r.v.s
Examples

Binary tree, branching at integer times

- $A(x) = x$: Branching random walk
- A step function: Derrida’s Generalised Random Energy models (GREM)
- Special case $A(x) = 0, x < 1, A(1) = 1$: Random energy model (REM), i.e. $n(t)$ iid $\mathcal{N}(0, t)$ r.v.s

Supercritical Galton-Watson tree
Examples

Binary tree, branching at integer times

- $A(x) = x$: Branching random walk
- A step function: Derrida’s Generalised Random Energy models (GREM)
- Special case $A(x) = 0, x < 1, A(1) = 1$: Random energy model (REM), i.e. $n(t)$ iid $\mathcal{N}(0, t)$ r.v.s

Supercritical Galton-Watson tree

- $A(x) = x$: Branching Brownian motion (BBM)
Examples

Binary tree, branching at integer times

- \( A(x) = x \): Branching random walk
- A step function: Derrida’s Generalised Random Energy models (GREM)
- Special case \( A(x) = 0, x < 1, A(1) = 1 \): Random energy model (REM), i.e. \( n(t) \) iid \( \mathcal{N}(0, t) \) r.v.s

Supercritical Galton-Watson tree

- \( A(x) = x \): Branching Brownian motion (BBM)
- General \( A \): variable speed BBM [Derrida-Spohn, Fang-Zeitouni]
Extreme value theory
Extreme value theory

In the class of models we have described, we are interested in three main questions:
Extreme value theory

In the class of models we have described, we are interested in three main questions:

- How big is $M(t) \equiv \max_{k \leq n(t)} x_k(t)$, as $t \uparrow \infty$?
Extreme value theory

In the class of models we have described, we are interested in three main questions:

- How big is $M(t) \equiv \max_{k \leq n(t)} x_k(t)$, as $t \uparrow \infty$?
- Is there a rescaling $u_t(x)$, such that
  \[ \mathbb{P}(M(t) \leq u_t(x)) \rightarrow F(x) \]
Extreme value theory

In the class of models we have described, we are interested in three main questions:

- How big is $M(t) \equiv \max_{k \leq n(t)} x_k(t)$, as $t \uparrow \infty$?
- Is there a rescaling $u_t(x)$, such that
  $$\mathbb{P}(M(t) \leq u_t(x)) \to F(x)?$$
- Is there a limiting extremal process, $\mathcal{P}$, such that
  $$\sum_{k \leq n(t)} \delta_{u_t^{-1}(x_k(t))} \to \mathcal{P}?$$
Reference: The REMs
Reference: The REMs

All well understood in the case of the REM. Then $x_k(t)$ are just $n(t)$ iid Gaussian rv’s with variance $t$. 

$M(t) \sim \sqrt{t} \sqrt{\frac{1}{2} \ln n(t)}$

With $u_t(x) = \sqrt{t} (\sqrt{2 \ln n(t)} - \ln \ln n(t)^2 + \sqrt{2 \ln n(t)})$, $P(M(t) \leq u_t(x)) \to \exp\left(-\frac{1}{4} \pi e^{-\sqrt{2} x} \right)$

and

$\sum_{k \leq n(t)} \delta u_{-1} t(x_k(t)) \to \text{PPP}\left(\frac{1}{4} \pi e^{-\sqrt{2} x} dx\right)$

where PPP($\mu$) denotes the Poisson Point Process with intensity $\mu$. 

A. Bovier (IAM Bonn)
Reference: The REMs

All well understood in the case of the REM. Then $x_k(t)$ are just $n(t)$ iid Gaussian rv’s with variance $t$.

- $M(t) \sim \sqrt{t} \sqrt{2 \ln n(t)}$
Reference: The REMs

All well understood in the case of the REM. Then $x_k(t)$ are just $n(t)$ iid Gaussian rv’s with variance $t$.

- $M(t) \sim \sqrt{t} \sqrt{2 \ln n(t)}$

- With $u_t(x) = \sqrt{t} \left( \sqrt{2 \ln n(t)} - \frac{\ln \ln n(t)}{2\sqrt{2 \ln n(t)}} + \frac{x}{\sqrt{\ln n(t)}} \right)$,

$$\mathbb{P}(M(t) \leq u_t(x)) \to \exp \left( -\frac{1}{4\pi} e^{-\sqrt{2}x} \right)$$
Gaussian processes on trees

Reference: The REMs

All well understood in the case of the REM. Then $x_k(t)$ are just $n(t)$ iid Gaussian rv’s with variance $t$.

- $M(t) \sim \sqrt{t} \sqrt{2 \ln n(t)}$
- With $u_t(x) = \sqrt{t} \left( \sqrt{2 \ln n(t)} - \frac{\ln \ln n(t)}{2\sqrt{2 \ln n(t)}} + \frac{x}{\sqrt{\ln n(t)}} \right)$,

$$\mathbb{P}(M(t) \leq u_t(x)) \to \exp \left( -\frac{1}{4\pi} e^{-\sqrt{2}x} \right)$$

and

$$\sum_{k \leq n(t)} \delta_{u_t^{-1}(x_k(t))} \to \text{PPP}(\frac{1}{4\pi} e^{-\sqrt{2}x} dx)$$

where PPP($\mu$) denotes the Poisson Point Process with intensity $\mu$. 

A. Bovier (IAM Bonn)
Universality 1: the order of the maximum
Universality 1: the order of the maximum

In all models, it has been shown (or can be shown easily) that the order of the maximum is only a function of the concave hull of the function $A$ (and on the growth rate of $n(t)$):
Universality 1: the order of the maximum

In all models, it has been shown (or can be shown easily) that the order of the maximum is only a function of the concave hull of the function $A$ (and on the growth rate of $n(t)$):

If $\bar{A}$ denotes the concave hull of $A$, then:

$$\lim_{t \to \infty} t^{-1} M(t) = \sqrt{2} \sqrt{\lim_{t \to \infty} t^{-1} \ln n(t)} \int_0^1 \sqrt{\frac{d}{ds} \bar{A}(s)} ds$$

[AB-I. Kurkova 01, for binary tree,
Universality 1: the order of the maximum

In all models, it has been shown (or can be shown easily) that the order of the maximum is only a function of the concave hull of the function $A$ (and on the growth rate of $n(t)$):

If $\bar{A}$ denotes the concave hull of $A$, then:

$$\lim_{t \to \infty} t^{-1} M(t) = \sqrt{2} \sqrt{\lim_{t \to \infty} t^{-1} \ln n(t)} \int_0^1 \sqrt{\frac{d}{ds} \bar{A}(s)} ds$$

[AB-I. Kurkova 01, for binary tree,

If $A(s) \leq s$, for all $s \leq 1$, then $\bar{A}(s) = s$ and the order of the maximum is the same as in the REM.
The GREM

If $A(s) < s$, for all $s \in (0, 1)$, then all results are the same as in the corresponding REM!

If $A(s) \leq s$, with equality in a finite number of points, the REM picture holds, but a prefactor appears in front of the $e^{-x}$'s.

If $\bar{A}(s) \neq s$, then the leading order and the logarithmic correction are changed and depend on $\bar{A}$; the extremal process is a Poisson cascade process.

Note the special role of the linear function $A(s) = s$. Bovier (IAM Bonn)
The GREM

The full picture is known (or easy to get) if \( A \) is a step function. In that case:

If \( A(s) < s \), for all \( s \in (0, 1) \), then all results are the same as in the corresponding REM!

If \( A(s) \leq s \), with equality in a finite number of points, the REM picture holds, but a prefactor appears in front of the \( e^{-x} \)'s.

If \( \bar{A}(s) \neq s \), then the leading order and the logarithmic correction are changed and depend on \( \bar{A} \); the extremal process is a Poisson cascade process.

Note the special role of the linear function \( A(s) = s \).
The GREM

The full picture is known (or easy to get) if $A$ is a step function. In that case:

- If $A(s) < s$, for all $s \in (0, 1)$, then all results are the same as in the corresponding REM!
The GREM

The full picture is known (or easy to get) if $A$ is a **step function**. In that case:

- If $A(s) < s$, for all $s \in (0, 1)$, then all results are the same as in the corresponding REM!
- If $A(s) \leq s$, with equality in a finite number of points, the REM picture holds, but a prefactor appears in front of the $e^{-x}$'s.

Note the special role of the linear function $A(s) = s$. 

A. Bovier (IAM Bonn)
The **GREM**

The full picture is known (or easy to get) if $A$ is a **step function**. In that case:

- If $A(s) < s$, for all $s \in (0, 1)$, then all results are the same as in the corresponding REM!
- If $A(s) \leq s$, with equality in a finite number of points, the REM picture holds, but a prefactor appears in front of the $e^{-x}$'s.
- If $\bar{A}(s) \neq s$, then the leading order and the logarithmic correction are changed and depend on $\bar{A}$; the extremal process is a **Poisson cascade process**.
The GREM

The full picture is known (or easy to get) if $A$ is a step function. In that case:

- If $A(s) < s$, for all $s \in (0, 1)$, then all results are the same as in the corresponding REM!
- If $A(s) \leq s$, with equality in a finite number of points, the REM picture holds, but a prefactor appears in front of the $e^{-x}$'s.
- If $\bar{A}(s) \neq s$, then the leading order and the logarithmic correction are changed and depend on $\bar{A}$; the extremal process is a Poisson cascade process.

Note the special role of the linear function $A(s) = s$
Branching Brownian motion

(BBM) is a classical object in probability, combining the standard models of **random motion** and **random genealogies** into one: Each particle of the Galton-Watson process performs Brownian motion independently of any other.
Branching Brownian motion

(BBM) is a classical object in probability, combining the standard models of random motion and random genealogies into one: Each particle of the Galton-Watson process performs Brownian motion independently of any other.

BBM is the canonical model of a spatial branching process.

Picture by Matt Roberts, Bath
The F-KPP

One of the simplest reaction-diffusion equations is the Fisher-Kolmogorov-Petrovsky-Piscounov (F-KPP) equation:

$$\frac{\partial v(x,t)}{\partial t} = \frac{1}{2} \frac{\partial^2 v(x,t)}{\partial x^2} + v(x,t) - v(x,t)^2$$

Fischer used this equation to model the evolution of biological populations. It accounts for:

- birth: $v(x,t)$,
- death: $-v(x,t)^2$,
- diffusive migration: $\frac{\partial^2 v(x,t)}{\partial x^2}$.
One of the simplest reaction-diffusion equations is the Fisher-Kolmogorov-Petrovsky-Piscounov (F-KPP) equation:

$$\partial_t v(x, t) = \frac{1}{2} \partial_x^2 v(x, t) + v - v^2$$
The F-KPP

One of the simplest reaction-diffusion equations is the Fisher-Kolmogorov-Petrovsky-Piscounov (F-KPP) equation:

$$\partial_t v(x, t) = \frac{1}{2} \partial_x^2 v(x, t) + v - v^2$$

Fischer used this equation to model the evolution of biological populations. It accounts for:

- **birth**: $v$,
- **death**: $-v^2$,
- **diffusive migration**: $\partial_x^2 v$. 

Let $f: \mathbb{R} \to [0,1]$ and \{ $x_k(\tau) : k \leq n(\tau)$ \} be BBM.

Then $v \equiv 1 - u$ is the solution of the F-KPP equation with initial condition $v(0, x) = 1 - f(x)$. 
F-KPP equation and BBM

Lemma (McKeane ’75, Ikeda, Nagasawa, Watanabe ’69)

Let $f : \mathbb{R} \to [0, 1]$ and $\{x_k(t) : k \leq n(t)\}$ BBM.

$$u(t, x) = \mathbb{E} \left[ \prod_{k=1}^{n(t)} f(x - x_k(t)) \right]$$

Then $v \equiv 1 - u$ is the solution of the F-KPP equation with initial condition $v(0, x) = 1 - f(x)$. 
Travelling waves

The equation

\[ \omega'' + \sqrt{2} \omega' - \omega^2 + \omega = 0. \]

has a unique solution satisfying

\[ 0 < \omega(x) < 1, \quad \omega(x) \to 0, \quad \text{as } x \to +\infty, \]

and

\[ \omega(x) \to 1, \quad \text{as } x \to -\infty, \]

up to translation, i.e. if \( \omega, \omega' \) are two solutions, then there exists an \( a \in \mathbb{R} \) s.t.

\[ \omega'(x) = \omega(x + a). \]

For suitable initial conditions,

\[ u(t, x + m(t)) \to \omega(x), \]

where

\[ m(t) = \sqrt{2}t - \frac{3}{2} \sqrt{2} \ln t, \]

where \( \omega \) is one of the stationary solutions.
Travelling waves

Theorem (Bramson ’78)

The equation

$$\frac{1}{2} \omega'' + \sqrt{2} \omega' - \omega^2 + \omega = 0.$$ 

has a unique solution satisfying $0 < \omega(x) < 1$, $\omega(x) \to 0$, as $x \to +\infty$, and $\omega(x) \to 1$, as $x \to -\infty$, up to translation, i.e. if $\omega, \omega'$ are two solutions, then there exists $a \in \mathbb{R}$ s.t. $\omega'(x) = \omega(x + a)$. 

For suitable initial conditions, $u(t, x + m(t)) \to \omega(x)$, where $m(t) = \sqrt{2} t - \frac{3}{2} \sqrt{2} \ln t$, where $\omega$ is one of the stationary solutions.
Travelling waves

Theorem (Bramson ’78)

The equation

\[ \frac{1}{2} \omega'' + \sqrt{2} \omega' - \omega^2 + \omega = 0. \]

has a unique solution satisfying 0 < \( \omega(x) < 1 \), \( \omega(x) \to 0 \), as \( x \to +\infty \), and \( \omega(x) \to 1 \), as \( x \to -\infty \), up to translation, i.e. if \( \omega, \omega' \) are two solutions, then there exists \( a \in \mathbb{R} \) s.t. \( \omega'(x) = \omega(x + a) \).

For suitable initial conditions,

\[ u(t, x + m(t)) \to \omega(x), \]

where \( m(t) = \sqrt{2t} - \frac{3}{2\sqrt{2}} \ln t \), where \( \omega \) is one of the stationary solutions.
Examples

This theorem allows to represent:

\[ u(t, x) = P(\max_{k \leq n}(t) x_k(t) \leq x), \]

and the Laplace functional

\[ u(t, x) = E\exp(-\sum_{k \leq n}(t) \phi(x_k(t))). \]

In particular, it gives Bramson's celebrated result that

\[ \lim_{t \to \infty} P(\max_{k \leq n}(t) x_k(t) - m(t) \leq x) = \omega(x). \]
Examples

This theorem allows to represent:

\[
u(t, x) = P(\max_{k \leq n(t)} x_k(t) \leq x),
\]

and the Laplace functional

\[
u(t, x) = E \exp\left(-\sum_{k \leq n(t)} \phi(x_k(t))\right).
\]

In particular, it gives Bramson's celebrated result that

\[
\lim_{t \to \infty} P\left(\max_{k \leq n(t)} x_k(t) - m(t) \leq x\right) = \omega(x).
\]
Examples

This theorem allows to represent:

\[ u(t, x) = \mathbb{P}(\max_{k \leq n(t)} x_k(t) \leq x) , \]
Examples

This theorem allows to represent:

- \( u(t, x) = \mathbb{P}(\max_{k \leq n(t)} x_k(t) \leq x) \), and

- the Laplace functional \( u(t, x) = \mathbb{E} \exp(- \sum_{k \leq n(t)} \phi(x_k(t))) \)
Examples

This theorem allows to represent:

- \( u(t, x) = \mathbb{P}(\max_{k \leq n(t)} x_k(t) \leq x) \), and

- the Laplace functional \( u(t, x) = \mathbb{E} \exp(-\sum_{k \leq n(t)} \phi(x_k(t))) \)

In particular, it gives Bramson’s celebrated result that

\[
\lim_{t \to \infty} \mathbb{P}(\max_{k \leq n(t)} x_k(t) - m(t) \leq x) = \omega(x)
\]
Lalley-Sellke representation

\[ \omega(x) \] is random shift of Gumbel-distribution

\[ \omega(x) = E \left[ e^{-CZ} \right], \]

\[ Z(d) = \lim_{t \to \infty} Z(t), \]

where \( Z(t) \) is the derivative martingale,

\[ Z(t) = \sum_{k \leq n(t)} \left\{ \sqrt{2t} - x_k(t) \right\} e^{-\sqrt{2} \left\{ \sqrt{2t} - x_k(t) \right\}} \]
Lalley-Sellke representation

Lalley-Sellke, 1987: \( \omega(x) \) is random shift of Gumbel-distribution

\[
\omega(x) = \mathbb{E} \left[ e^{-CZ} e^{-\sqrt{2}x} \right],
\]

where \( Z(t) \) is the derivative martingale,

\[
Z(t) = \sum_{k \leq n(t)} \left\{ \sqrt{2} t - x_k(t) \right\} e^{-\sqrt{2} \left\{ \sqrt{2} t - x_k(t) \right\}}.
\]
Lalley-Sellke representation

Lalley-Sellke, 1987: \( \omega(x) \) is random shift of Gumbel-distribution

\[
\omega(x) = \mathbb{E} \left[ e^{-CZ} e^{-\sqrt{2x}} \right],
\]

\( Z \overset{(d)}{=} \lim_{t \to \infty} Z(t) \), where \( Z(t) \) is the derivative martingale,

\[
Z(t) = \sum_{k \leq n(t)} \left\{ \sqrt{2} t - x_k(t) \right\} e^{-\sqrt{2} \left( \sqrt{2} t - x_k(t) \right)}
\]
Lalley-Sellke representation

Lalley-Sellke, 1987: \( \omega(x) \) is random shift of Gumbel-distribution

\[
\omega(x) = \mathbb{E} \left[ e^{-CZ} e^{-\sqrt{2}x} \right],
\]

\( Z \overset{(d)}{=} \lim_{t \to \infty} Z(t) \), where \( Z(t) \) is the derivative martingale,

\[
Z(t) = \sum_{k \leq n(t)} \left\{ \sqrt{2}t - x_k(t) \right\} e^{-\sqrt{2}\left\{ \sqrt{2}t - x_k(t) \right\}}
\]
The extremal process of BBM

The extremal process

\[ E_t \equiv n(t) \sum_{i=1}^{\delta x_i(t) - m(t)}. \]

Poisson Point Process: \[ P_{Z} = \sum_{i \in N} \delta p_i \equiv PPP (CZe - \sqrt{2} x dx) \]

Cluster process: \[ \Delta(t) \equiv \sum k \delta x_k(t) - \max_{j \leq n(t)} x_j(t). \]

conditioned on the event \[ \{ \max_{j \leq n(t)} x_j(t) > \sqrt{2} t \} \]

converges in law to point process, \( \Delta \). [Chauvin, Rouault '90]

A. Bovier (IAM Bonn)
The extremal process

\[ \mathcal{E}_t \equiv \sum_{i=1}^{n(t)} \delta_{x_i(t) - m(t)}. \]

Poisson Point Process: Set

\[ P_Z = \sum_{i \in \mathbb{N}} \delta_{p_i} \equiv \text{PPP} \left( CZe^{-\sqrt{2}x} \, dx \right). \]
The extremal process

\[ \mathcal{E}_t \equiv \sum_{i=1}^{n(t)} \delta_{x_i(t)} - m(t). \]

Poisson Point Process: Set

\[ \mathcal{P}_Z = \sum_{i \in \mathbb{N}} \delta_{p_i} \equiv \text{PPP} \left( C Ze^{-\sqrt{2}x} \, dx \right) \]

Cluster process:

\[ \Delta(t) \equiv \sum_k \delta_{x_k(t)} - \max_{j \leq n(t)} x_j(t). \]

conditioned on the event \( \{ \max_{j \leq n(t)} x_j(t) > \sqrt{2}t \} \) converges in law to point process, \( \Delta \). [Chauvin, Rouault ’90]
The extremal process

Theorem (Arguin-B-Kistler '11, Aidékon, Brunet, Berestycki, Shi '11)

With the notation above, the point process \( \mathcal{E}_t \) converges in law to a point process \( \mathcal{E} \), given by

\[
\mathcal{E} \equiv \sum_{i, j \in \mathbb{N}} \delta_{p_i + \Delta(i) - j}
\]
The extremal process

Theorem (Arguin-B-Kistler '11, Aidékon, Brunet, Berestycki, Shi '11)

With the notation above, the point process $\mathcal{E}_t$ converges in law to a point process $\mathcal{E}$, given by

$$\mathcal{E} \equiv \sum_{i,j \in \mathbb{N}} \delta_{p_i + \Delta_j^{(i)}}$$
The extremal process

**Theorem (Arguin-B-Kistler ’11, Aidékon, Brunet, Berestycki, Shi ’11)**

*With the notation above, the point process $\mathcal{E}_t$ converges in law to a point process $\mathcal{E}$, given by*

$$\mathcal{E} \equiv \sum_{i,j \in \mathbb{N}} \delta_{p_i + \Delta_j^{(i)}}$$
The extremal process of BBM

The extremal process
The extremal process

Technically, proven by showing convergence of Laplace functionals:

$$E\left[ \exp\left( -\int \phi(y) E_t(dy) \right) \right] \to E\left[ \exp\left( -C(\phi) Z \right) \right]$$

for any $\phi \in C_c(\mathbb{R})$ non-negative, where

$$C(\phi) = \lim_{t \to \infty} \sqrt{2\pi} \int_0^\infty \left( 1 - u(t, y + \sqrt{2t}) \right) y e^{\sqrt{2t}y} dy$$

$u(t, y)$: solution of F-KPP with initial condition $u(0, y) = e^{-\phi(y)}$.

Then show that the limit is the Laplace functional of the process $E$ described above.
The extremal process

Technically, proven by showing convergence of Laplace functionals:

\[
\mathbb{E} \left[ \exp \left( - \int \phi(y) \mathcal{E}_t(dy) \right) \right] \rightarrow \mathbb{E} \left[ \exp \left( -C(\phi) Z \right) \right]
\]

for any \( \phi \in C_c(\mathbb{R}) \) non-negative, where

\[
C(\phi) = \lim_{t \to \infty} \sqrt{\frac{2}{\pi}} \int_0^\infty \left( 1 - u(t, y + \sqrt{2}t) \right) ye^{\sqrt{2}y} dy
\]

\( u(t, y) \): solution of F-KPP with initial condition \( u(0, y) = e^{-\phi(y)} \).
The extremal process

Technically, proven by showing convergence of Laplace functionals:

$$E \left[ \exp \left( - \int \phi(y) \mathcal{E}_t(dy) \right) \right] \rightarrow E \left[ \exp \left( - C(\phi) Z \right) \right]$$

for any $\phi \in C_c(\mathbb{R})$ non-negative, where

$$C(\phi) = \lim_{t \to \infty} \sqrt{\frac{2}{\pi}} \int_0^\infty \left( 1 - u(t, y + \sqrt{2}t) \right) ye^{\sqrt{2}y} dy$$

$u(t, y)$: solution of F-KPP with initial condition $u(0, y) = e^{-\phi(y)}$.

Then show that the limit is the Laplace functional of the process $\mathcal{E}$ described above.
Variable speed BBM.....below the straight line...

Theorem (AB, L. Hartung '13,'14)

Assume that $A(x) < x$, $\forall x \in (0, 1)$, $A'(0) = a < 1$, $A'(1) = b > 1$.

Then $\exists C(b)$ and a r.v. $Y$ such that

$$P(M(t) - \tilde{m}(t) \leq x) \to Ee^{-C(b)Y} - \sqrt{2x} \sum_{k \leq n(t)} \delta x_k(t) - \tilde{m}(t) \to E a, b = \sum_{i, j} \delta p_i + b \Delta(i, j) \tilde{m}(t) \equiv \sqrt{2t - \frac{1}{2} \sqrt{2} \ln t}.$$

$p_i$: the atoms of a PPP($C(b)$), $\Delta$: as in BBM but with the conditioning on the event $\{\max_k x_k(t) \geq \sqrt{2b}t\}$. 

A. Bovier (IAM Bonn) Extremal Processes of Gaussian Processes Indexed by Trees
Theorem (AB, L. Hartung ’13,’14)

Assume that $A(x) < x$, $\forall x \in (0, 1)$, $A'(0) = a^2 < 1$, $A'(1) = b^2 > 1$. 
Variable speed BBM.....below the straight line...

Theorem (AB, L. Hartung ’13,’14)

Assume that $A(x) < x, \forall x \in (0, 1)$, $A'(0) = a^2 < 1$, $A'(1) = b^2 > 1$.

Then $\exists C(b)$ and a r.v. $Y_a$ such that
Theorem (AB, L. Hartung '13,'14)

Assume that $A(x) < x$, $\forall x \in (0, 1)$, $A'(0) = a^2 < 1$, $A'(1) = b^2 > 1$. Then $\exists C(b)$ and a r.v. $Y_a$ such that

$$\mathbb{P}(M(t) - \tilde{m}(t) \leq x) \to \mathbb{E}e^{-C(b)Y_a}e^{-\sqrt{2}x}$$
Variable speed BBM.....below the straight line...

**Theorem (AB, L. Hartung '13,’14)**

Assume that $A(x) < x$, $\forall x \in (0, 1)$, $A'(0) = a^2 < 1$, $A'(1) = b^2 > 1$.

Then $\exists C(b)$ and a r.v. $Y_a$ such that

1. $\mathbb{P} (\tilde{M}(t) - \tilde{m}(t) \leq x) \rightarrow \mathbb{E} e^{-C(b)Y_a}e^{-\sqrt{2x}}$

2. $\sum_{k \leq n(t)} \delta x_k(t) - \tilde{m}(t) \rightarrow E_{a,b} = \sum_{i,j} \delta p_i + b\Delta_j^{(i)}$
Variable speed BBM.....below the straight line...

**Theorem (AB, L. Hartung ’13,’14)**

Assume that $A(x) < x, \forall x \in (0, 1)$, $A'(0) = a^2 < 1$, $A'(1) = b^2 > 1$. Then $\exists C(b)$ and a r.v. $Y_a$ such that

- $\mathbb{P}(M(t) - \tilde{m}(t) \leq x) \to \mathbb{E}e^{-C(b)Y_a e^{-\sqrt{2}x}}$
- $\sum_{k \leq n(t)} \delta x_k(t) - \tilde{m}(t) \to \mathcal{E}_{a,b} = \sum_{i,j} \delta p_i + b\Delta^{(i)}$

$\tilde{m}(t) \equiv \sqrt{2}t - \frac{1}{2\sqrt{2}} \ln t$.

$p_i$: e the atoms of a PPP$(C(b) Y_a e^{-\sqrt{2}x} dx)$,

$\Delta$: are as in BBM but with the conditioning on the event $\{\max_k x_k(t) \geq \sqrt{2}bt\}$. 
Elements of the proof:

1) Explicit construction for the case of two speeds:

\[ A(s) = \begin{cases} 
σ_2^2 & \text{if } 0 < b \leq 1 \\
σ_1^2 + (s - b)σ_2^2 & \text{if } bt \leq s \leq t, \\
0 & \text{otherwise.} 
\end{cases} \]

There are three major steps needed beyond those in standard BBM:

- Localisation of the particles reaching extreme levels at the time of the speed change in a narrow (\(\sqrt{t}\)) gate around \(\sqrt{2}bt\).

Proof of uniform integrability of the McKean martingale

\[ Y_s \equiv \sum_{n(s)} e^{-(1+σ^2_1)s + \sqrt{2}x_i(n(s))}, \]

Asymptotics of solutions of the FKPP equation at very large values ahead of the travelling wave.
Elements of the proof:

1) Explicit construction for the case of two speeds:

\[
A(s) = \begin{cases} 
  s\sigma_1^2 & 0 \leq s < bt \\
  b\sigma_1^2 + (s - b)\sigma_2^2 & bt \leq s \leq t 
\end{cases}, \quad 0 < b \leq 1.
\]
Elements of the proof:

1) Explicit construction for the case of two speeds:

\[ A(s) = \begin{cases} 
 s \sigma_1^2 & 0 \leq s < bt \\
 b \sigma_1^2 + (s - b) \sigma_2^2 & bt \leq s \leq t 
\end{cases}, \quad 0 < b \leq 1. \]

There are three major steps needed beyond those in standard BBM:
Elements of the proof:

1) Explicit construction for the case of two speeds:

\[ A(s) = \begin{cases} 
    s\sigma^2_1 & 0 \leq s < bt \\
    b\sigma^2_1 + (s - b)\sigma^2_2 & bt \leq s \leq t 
\end{cases}, \quad 0 < b \leq 1. \]

There are three major steps needed beyond those in standard BBM:

- Localisation of the particles reaching extreme levels at the time of the speed change in a narrow \((\sqrt{t})\) gate around \(\sqrt{2bt\sigma^2_1} = tA(b)\).
Elements of the proof:

1) Explicit construction for the case of two speeds:

\[ A(s) = \begin{cases} 
  s\sigma_1^2 & 0 \leq s < bt \\
  b\sigma_1^2 + (s - b)\sigma_2^2 & bt \leq s \leq t 
\end{cases} , \quad 0 < b \leq 1. \]

There are three major steps needed beyond those in standard BBM:

- Localisation of the particles reaching extreme levels at the time of the speed change in a narrow \((\sqrt{t})\) gate around \(\sqrt{2bt}\sigma_1^2 = tA(b)\).
- Proof of uniform integrability of the McKean martingale

\[ Y_s \equiv \sum_{i=1}^{n(s)} e^{-s(1+\sigma_1^2)+\sqrt{2x_i(s)}} \]
Elements of the proof:

1) Explicit construction for the case of two speeds:

\[ A(s) = \begin{cases} 
  s\sigma_1^2 & 0 \leq s < bt \\
  b\sigma_1^2 + (s - b)\sigma_2^2 & bt \leq s \leq t
\end{cases}, \quad 0 < b \leq 1. \]

There are three major steps needed beyond those in standard BBM:

- Localisation of the particles reaching extreme levels at the time of the speed change in a narrow \((\sqrt{t})\) gate around \(\sqrt{2}bt\sigma_1^2 = tA(b)\).
- Proof of uniform integrability of the McKean martingale
  \[ Y_s \equiv \sum_{i=1}^{n(s)} e^{-s(1+\sigma_1^2)+\sqrt{2}x_i(s)} \]
- Asymptotics of solutions of the FKPP equation at very large values ahead of the travelling wave.
Elements of the proof:
Elements of the proof:

2) Gaussian comparison for general $A$:

Use comparison for Laplace functionals with two-speed process; only good approximation of covariance near 0 and 1 needed.
Variable speed BBM

Above the straight line

When the concave hull of $A$ is above the straight line, everything changes. If $A$ is piecewise linear, it is quite easy to get the full picture: Cascade of BBM processes.

If $A$ is strictly concave, Fang and Zeitouni '12 and Maillard and Zeitouni '13 have shown that the correct rescaling is

$$m(t) = C_\sigma t^{1/3} - D_\sigma t^{2} (1) \ln t$$

(with explicit constants $C_\sigma$ and $D_\sigma$) but there are no explicit limit laws or limit processes available.
Above the straight line

When the concave hull of $A$ is above the straight line, everything changes.
Above the straight line

When the concave hull of $A$ is above the straight line, everything changes.

- If $A$ is **piecewise linear**, it is quite easy to get the full picture:
  
  Cascade of BBM processes.
Above the straight line

When the concave hull of $A$ is above the straight line, everything changes.

- If $A$ is \textit{piecewise linear}, it is quite easy to get the full picture:
  Cascade of BBM processes.

- If $A$ is strictly concave, Fang and Zeitouni '12 and Maillard and Zeitouni '13 have shown that the correct rescaling is
  \[
  m(t) = C_\sigma t - D_\sigma t^{1/3} - \sigma^2(1) \ln t
  \]
  (with explicit constants $C_\sigma$ and $D_\sigma$) but there are no explicit limit laws or limit processes available.
Universality

The new extremal processes should not be limited to BBM: Branching random walk [Bramson, Addario-Berry, A´ıdekon (law of max), Madaule '13 (full extremal process),...].

Gaussian free field in $d = 2$ [Bolthausen, Deuschel, Giacomin, Bramson-Ding-Zeitouni, Biskup-Louidor '13 [Poisson cluster extremes] ....]

Cover times of random walks [Sznitman, Dembo-Peres-Rosen-Zeitouni, Belius, Belius-Kistler ....]

Spin glasses with log-correlated potentials [Fyodorov, Bouchaud,..]

............

A. Bovier (IAM Bonn)
Universality

The **new extremal processes** should not be limited to BBM:
Universality

The new extremal processes should not be limited to BBM:

- **Branching random walk** [Bramson, Addario-Berry, Aïdékon (law of max), Madaule ’13 (full extremal process),...]
- **Gaussian free field in** $d = 2$ [Bolthausen, Deuschel, Giacomin, Bramson-Ding-Zeitouni, Biskup-Louidor ’13 [Poisson cluster extremes] ....]
Universality

The new extremal processes should not be limited to BBM:

- **Branching random walk** [Bramson, Addario-Berry, Aïdékon (law of max), Madaule ’13 (full extremal process),...]
- **Gaussian free field in** $d = 2$ [Bolthausen, Deuschel, Giacomin, Bramson-Ding-Zeitouni, Biskup-Louidor ’13 [Poisson cluster extremes] ....]
- **Cover times of random walks** [Sznitman, Dembo-Peres-Rosen-Zeitouni, Belius, Belius-Kistler ....]
Universality

The new extremal processes should not be limited to BBM:

- **Branching random walk** [Bramson, Addario-Berry, Aïdékon (law of max), Madaule ’13 (full extremal process),...]

- **Gaussian free field in** $d = 2$ [Bolthausen, Deuschel, Giacomin, Bramson-Ding-Zeitouni, Biskup-Louidor ’13 [Poisson cluster extremes] ....]

- **Cover times** of random walks [Sznitman, Dembo-Peres-Rosen-Zeitouni, Belius, Belius-Kistler ....]

- **Spin glasses** with log-correlated potentials [Fyodorov, Bouchaud,..]

- .............
Thank you for your attention!