Geometric Tomography and Harmonic Analysis

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Geometric Tomography is the area of Mathematics where one investigates properties of solids based on the information about their sections and projections. It shares ideas and methods from many fields of Mathematics, such as Differential Geometry, Functional Analysis, Harmonic Analysis, Combinatorics and Probability. But the most significant overlap is with Convex Geometry and in particular with the classical Brunn-Minkowski theory. The workshop brought together a number of top researchers as well as students and postdocs with the aim of discussing most recent developments in the area.

The topics of the workshop included harmonic analysis on the sphere, spherical operators and special classes of bodies, geometric inequalities, discrete geometry, probability and random matrices.

We start the description with a harmonic analysis type result proved by Wolfgang Weil and his collaborators. Goodey, Yaskin and Yaskina in 2009 introduced certain operators I_p on $C^\infty(S^{n-1})$ by means of the Fourier transform. These operators turned out to be useful for various applications. Goodey and Weil used these operators I_p for the study of k-th mean section bodies. Informally speaking, $M_k(K)$, the k-the mean section body of a convex body K in $\mathbb{R}^n$ is the Minkowski sum of all its sections by k-dimensional affine planes. Until recently it was unknown (up to some particular cases) whether convex bodies are uniquely determined by their k-th mean section bodies. Goodey and Weil proved that this is indeed true. Moreover, they established an interesting connection between the support function of $M_k(K)$ and the $(n+1-k)$-th surface area measure of K, which involves a combination of operators I_p mentioned previously. Further research in this direction led Goodey, Hug and Weil to a new Crofton-type formula for area measures and also to a local version of the Principal Kinematic Formula.

Mathieu Meyer presented recent results obtained in collaboration with Carsten Schütt and Elisabeth Werner on affine points. An *affine invariant point* on the class of convex bodies $\mathcal{K}^n$ in $\mathbb{R}^n$, endowed with the Hausdorff metric, is a continuous map from $\mathcal{K}^n$ to $\mathbb{R}^n$ which is invariant under one-to-one affine transformations $\mathbb{R}^n$. This concept was introduced by Grünbaum. Well known examples of affine invariant points are the centroid and the Santalo point.

Earlier the authors established a number of results about affine points. In particular, they answered in the negative a question by Grünbaum who asked if there exists a finite basis of affine invariant points. They gave a positive answer to another question by Grünbaum about the size of the set of all affine invariant points.

More recently, the authors introduced a new concept of a dual affine point q of an affine invariant point p. It is defined by the formula $q(K^{p(K)}) = p(K)$ for every $K \in \mathcal{K}^n$, where $K^{p(K)}$ denotes the polar of K with respect to the point $p(K)$.

The authors answered a number of questions. In particular, does every affine invariant point p have a dual? The answer is negative. They also show that if a dual affine invariant point exists, it is unique. A reflexivity principle for affine invariant points is established, namely that the double dual point of p equals p. Furthermore, new examples of affine invariant points are presented.

Vitali Milman gave a talk on characterizing summations of convex sets. Details of the proof were presented by Liran Rotem in his subsequent lecture. This work continues a series of studies of fundamental concepts in convex geometry by means of the corresponding characterization theorems. It is a classical result of Minkowski that volume is polynomial with respect to the Minkowski addition.

One can consider other summations, in place of the Minkowski summation. One of the important examples is the p-summation: $h_{A+_pB} = (h_A^p + h_B^p)^{1/p}$, where h is the corresponding support function. Each summation $\bigoplus$ defines an induced homothety $\bigodot$ as follows: for $m \in \mathbb{N}$,

$$m \bigodot A = A \bigoplus \cdots \bigoplus A \quad (m \text{ times}).$$

Two of the main questions are the following. Does the homothety recovers the summation? What sums have the polynomiality property for the volume functional? Answers to these questions were given in the talk. If there is a function $f : \mathbb{N} \to \mathbb{R}_+$ such that $m \bigodot A = f(m)A$, then certain conditions guarantee when the corresponding sum is a corresponding p-sum.

It is also shown that if the summation is polynomial (and satisfies certain natural conditions), then the summation is either the Minkowski summation or the ∞-summation that is defined by $A +_\infty B = \text{conv}(A \cup B)$.

Liran Rotem gave a talk on "Isotropicity with respect to a measure". Recall that a measure μ is called *isotropic* if $\int_{\mathbb{R}^n} \langle x, \theta \rangle^2 d\mu(x)$ is independent of the unit vector θ. For every measure μ there exists a linear transformation $T \in SL(n)$ such that the push-forward of μ by T is isotropic. If C is a body in $\mathbb{R}^n$, define $\mu_C(A) = \mu(A \cap C)$. C is said to be isotropic with respect to μ if μ_C is isotropic.

The following questions are studied:

1) Given C and μ, does there exist $T \in SL(n)$ such that TC is isotropic with respect to μ.

2) If "yes", which properties will C have in such a position?

Regarding question 1, it was shown earlier by Bobkov that this is true in the case of the Gaussian measure on $\mathbb{R}^n$.

The following result is obtained: TC is isotropic with respect to a rotation invariant measure μ if and only if T is a critical point of a certain functional J. Some other related questions were also discussed.

Franz Schuster spoke about Cosine and Radon Transforms in the Theory of Minkowski Valuations.

Let $\mathcal{K}^n$ be the set of convex bodies in an $\mathbb{R}^n$. A map $\phi : \mathcal{K}^n \to \mathbb{R}$ is called a (real valued) *valuation* if

$$\phi(K) + \phi(L) = \phi(K \cup L) + \phi(K \cap L)$$

whenever $K,\ L,\ K \cup L \in \mathcal{K}^n$.

Valuations on convex bodies have been actively studied. A famous classical result in this area is Hadwiger's classification of rigid motion invariant real valued continuous valuations as linear combinations of the intrinsic volumes.

A map $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ is called a Minkowski valuation if

$$\Phi(K) + \Phi(L) = \Phi(K \cup L) + \Phi(K \cap L)$$

whenever $K,\ L,\ K \cup L \in \mathcal{K}^n$.

A problem is to describe the set $\mathrm{MVal}^{SO(n)}$ of continuous Minkowski valuations that are translation invariant and $SO(n)$-equivariant.

Open problem 1. For $\Phi \in \mathrm{MVal}^{SO(n)}$, are there $\Phi_i \in \mathrm{MVal}_i^{SO(n)}$ such that

$$\Phi = \Phi_0 + \Phi_1 + \cdots + \Phi_n?$$

Here, $\mathrm{MVal}_i^{SO(n)} = \{\Phi \in \mathrm{MVal}^{SO(n)} : \Phi(\lambda K) = \lambda^i \Phi(K)\}$.

Open problem 2. Find a classification/description of $\mathrm{MVal}_i^{SO(n)}$.

Regarding the description of $\mathrm{MVal}_i^{SO(n),\infty}$ (there is some natural notion of smoothness), Schuster has show that if $\Phi_i \in \mathrm{MVal}_i^{SO(n),\infty}$, $2 \le i \le n-2$, is even, then there is a unique smooth $O(i) \times O(n-i)$-invariant measure μ on the sphere such that

$$h(\Phi_i(K), \cdot) = \mathrm{vol}_i(K|\cdot) * \mu.$$

The measure μ is called the Crofton measure of Φ_i.

Jointly with Wannerer, Schuster proved that if a measure μ is the Crofton measure of an even $\Phi_i \in \mathrm{MVal}_i^{SO(n),\infty}$, then there exists $L \in \mathcal{K}^n$ such that

$$(C\mu)(u) = \int_{S^{n-1}} |\langle u, v\rangle| d\mu(v), \qquad u \in S^{n-1}.$$

They also obtained the following description. If $\Phi_i \in \mathrm{MVal}_i^{SO(n),\infty}$, $1 \le i \le n-1$, is even, then there exists a unique function $g \in C^\infty([-1, 1])$ such that

$$h(\Phi_i K, u) = \int_{S^{n-1}} g(\langle u, v\rangle) dS_i(K, u), \qquad u \in S^{n-1}.$$

The function g is called the generating function of Φ_i.

They also obtained a relation between μ and g in terms of the Radon transform on the Grassmanian and the inverse spherical cosine transform.

Astrid Berg presented a joint work with L. Parapatits, F. Schuster and M. Weberndorfer titled "Log-concavity properties of Minkowski valuations".

The classical Brunn-Minkowski inequality for intrinsic volumes asserts that for convex bodies K and L in $\mathbb{R}^n$, $1 < i \leq n$ and $0 < \lambda < 1$, the following holds

$$V_i((1-\lambda)K + \lambda L) \geq V_i(K)^{1-\lambda} V_i(L)^{\lambda}.$$

Let $MVal_j$ denote the class of continuous, translation invariant, $SO(n)$-equivariant and j-homogeneous Minkowski valuations. The authors studied the following question. Fix $\Phi_j \in MVal_j$. Does the family of inequalities

$$V_i(\Phi_j((1-\lambda)K + \lambda L)) \geq V_i(\Phi_j K)^{1-\lambda} V_i(\Phi_j L)^{\lambda}$$

hold? For which i does it hold?

The authors proved the following.

Theorem 0.1 *Let $1 \leq i \leq n$ and let $\Phi_j \in MVal_{j,i-1}$, $2 \leq j \leq n-1$, be non-trivial. If K and L are convex bodies (with non-empty interior), then for $\lambda \in (0,1)$,*

$$V_i(\Phi_j((1-\lambda)K + \lambda L)) \geq V_i(\Phi_j K)^{1-\lambda} V_i(\Phi_j L)^{\lambda},$$

with equality if and only if K and L are translate of each other.

Christos Saroglou gave a talk titled "Remarks on the conjectured log-Brunn-Minkowski inequality".

Böröczky, Lutwak, Yang and Zhang recently conjectured a certain strengthening of the Brunn-Minkowski inequality for symmetric convex bodies, the so-called log-Brunn-Minkowski inequality.

Conjecture. If K and L are centrally symmetric convex bodies, then for $\lambda \in (0,1)$,

$$V(\lambda K +_0 (1-\lambda)L) \geq V(K)^{\lambda} V(L)^{1-\lambda}.$$

Here

$$\lambda K +_0 (1-\lambda)L = \{x \in \mathbb{R}^n : \langle x, u \rangle \leq h_K^{\lambda}(u) h_L^{1-\lambda}(u),\ \forall u \in S^{n-1}\}.$$

The author proved the following result.

Theorem 0.2 *Let K and L be unconditional convex bodies (with respect to the same orthonormal basis) in $\mathbb{R}^n$ and $\lambda \in (0,1)$. Then*

$$V(\lambda K +_0 (1-\lambda)L) \geq V(K)^{\lambda} V(L)^{1-\lambda}.$$

Equality holds in the following case: whenever $K = K_1 \times \cdots \times K_m$, for some irreducible unconditional convex sets K_1,..., K_m, then there exist positive numbers such that $L = c_1 K_1 \times \cdots \times c_m K_m$.

Applications of this result are discussed. Moreover, it is shown that the log-Brunn-Minkowski inequality is equivalent to the (B)-Theorem for the uniform measure of the cube (this has been proven by Cordero-Erasquin, Fradelizi and Maurey for the gaussian measure instead).

Florian Besau presented his joint work with Franz Schuster on Binary Operations in Spherical Convex Geometry. This is in some respect related to Vitali Milman's talk since the goal is similar: to characterize binary operations. There is a result of Gardner, Hug and Weil that characterizes the Minkowski addition among binary operations on $\mathcal{K}^n$. The geometry of spherical convex sets is much less understood. Is there an analogue of the Minkowski addition on the sphere?

The authors define the operation of projection of a convex set on the sphere. They also define the spherical support function, using a geometric construction analogous to that in the Euclidean space.

Let $K^p_u(S^n)$ be the set of spherical convex bodies contained in S^+_u (the open hemisphere with centre u). Consider $K^p(S^n) = \cup_{u\in S^n} K^p_u(S^n)$.

Let $u \in S^n$. An operation $* : K^p_u(S^n) \times K^p_u(S^n) \to K^p_u(S^n)$ is called u-projection covariant if for all k-spheres S, $0 \le k \le n$ with $u \in S$, we have

$$K|S * L|S = (K * L)|S$$

for all $K,\, L \in K^p_u(S^n)$.

An operation $* : K^p(S^n) \times K^p(S^n) \to K^p(S^n)$ is called projection covariant if it is u-projection covariant for all $u \in S^n$.

The authors proved the following results.

Theorem 0.3 *Let $C = \cup_{u\in S^n}(K^p_u(S^n) \to K^p_u(S^n))$. An operation $* : C \to K^p(S^n)$ is projection covariant if and only if*

$$K * L = \mathrm{conv}(K \cup L) \quad \text{or} \quad K * L = -\mathrm{conv}(K \cup L)$$

or it is trivial (i.e. projection on the first or the second component, maybe with a minus sign).

Theorem 0.4 *An operation $* : K^p(S^n) \times K^p(S^n) \to K^p(S^n)$ is projection invariant and continuous if and only if it is trivial.*

Several presenters talked about new developments in the theory of random matrices. In particular, Susanna Spektor gave a talk based on the joint work with Alexander Litvak on Quantitative version of a Silverstein's result.

Let $w_{i,j}$ be i.i.d. copies of a random variable w with $\mathbb{E}w = 0$ and $\mathbb{E}w^2 = 1$. Consider the $p \times n$ matrix

$$W_n = \{w_{i,j}\}_{i\le p, j\le n}, \quad p = p(n),$$

and its sample covariance matrix

$$\Gamma_n = \frac{1}{n} W_n W_n^T.$$

The rows of W_n are denoted by X_i, $i \le p$.

A theorem of Silverstein asserts the following. Assuming that $p(n)/n \to c > 0$ as $n \to \infty$, $\|\Gamma_n\|_{op}$ converges in probability to a non-random quantity (which is $(1+\sqrt{c})^2$) if and only of $n^4\mathbb{P}(|w| \ge n) = o(1)$.

The authors' goal was to quantify Silverstein's result. Namely, they proved the following:

Theorem 0.5 *Let $\alpha \ge 2$, $c_0 > 0$. Let w be a random variable satisfying $\mathbb{E}w = 0$ and $\mathbb{E}w^2 = 1$ and*

$$\forall t \ge 1, \qquad \mathbb{P}(|w| \ge t) \ge \frac{c_0}{t^\alpha}.$$

Let W_n, X_j, Γ_n be as above. Then for every $K \ge 1$,

$$\mathbb{P}(\|\Gamma_n\|_{op} \ge K) \ge \mathbb{P}(\max_{i \le p} |X_i| \ge \sqrt{Kn}) \ge \min\left\{\frac{c_0 p}{4n^{(\alpha-2)/2}K^{\alpha/2}}, \frac{1}{2}\right\}.$$

Another talk on random matrices was given by Alexander Litvak, with the title "Approximating the covariance matrix with heavy tailed columns and RIP", joint work with O. Guédon, A. Pajor, N. Tomczak-Jaegermann.

Let A be a matrix whose columns $X_1, \ldots, X_N$ are independent random vectors in $\mathbb{R}^n$. Assume that p-th moments of $\langle X_i, a\rangle$, $a \in S^{n-1}$, $i \le N$, are uniformly bounded. For $p > 4$ the authors proved that with high probability A has the Restricted Isometry Property (RIP) provided that Euclidean norms $|X_i|$ are concentrated around $\sqrt{n}$ and that the covariance matrix is well approximated by the empirical covariance matrix provided that

$$\max_i |X_i| \le C(nN)^{1/4}.$$

They also provided estimates for RIP when

$$\mathbb{E}\,\phi\left(|\langle X_i, a\rangle|\right) \le 1$$

for $\phi(t) = (1/2)\exp(t^\alpha)$, with $\alpha \in (0, 2]$.

The theme of random matrices was continued in the talk "Small ball probabilities for linear images of high dimensional distributions" by Mark Rudelson (joint work with Roman Vershynin)

The main object of the study is a random vector $X = (X_1, ..., X_n)$ in $\mathbb{R}^n$ with independent coordinates X_i. Given a fixed $m \times n$ matrix A, the authors study the concentration properties of the random vector AX. They establish results of the following type:

If the distributions of X_i are well spread on the line, then the distribution of AX is well spread in space.

In particular, they extend the bound obtained by Paouris for log-concave measures to random vectors with independent coordinates.

Theorem 0.6 *Let $X = (X_1, ..., X_n) \in \mathbb{R}^n$ be a vector with independent coordinates and let the densities of X_i be bounded by K. If A is an $m \times n$ matrix, then*

$$\mathcal{L}(AX, t\|A\|_{HS}) \le \left(\frac{CKt}{\sqrt{\epsilon}}\right)^{(1-\epsilon)r(A)}, \qquad \textit{for all} t, \epsilon > 0.$$

Here, $r(A)$ is the stable rank of A, and $\mathcal{L}$ is the Levy concentration function.

Eric Grinberg gave a talk on "The Torus Transform on Symmetric Spaces of Compact Type", joint work with Steven Jackson. In a 1913 paper Paul Funk proved that a suitable function on the sphere S^2 is odd if and only if its integrals over great circles (closed geodesics) vanish, and that an even function is determined by such integrals. The authors replace the sphere S^2 by a symmetric space of compact type, e.g. a Grassmann manifold, and great circles by maximal totally geodesic at tori, and consider the transform that integrates over these. They showed that, when the symmetric space is the "universal covered space" in its class, the torus transform is injective, and otherwise the transform is non-injective, with a kernel that is directly linked to deck transformations of the appropriate symmetric cover. This gives one of the direct extensions of Funk's transform and its injectivity properties.

The talk of Yves Martinez-Maure focused on the question "Can hedgehogs be useful for Geometric Tomography?" Hedgehogs are geometrical objects that describe the Minkowski differences of arbitrary convex bodies in $\mathbb{R}^n$. For a function $h \in C^2(S^{n-1})$ the corresponding hedgehog can be parameterized as

$$x_h = h(u)u + \nabla h(u), \quad u \in S^{n-1}.$$

One can extend many classical notions to hedgehogs, such as its curvature function R_h, algebraic area, algebraic volume, projection hedgehog, hedgehog of constant width etc.

One of the interesting (and powerful) applications of hedgehogs to convex geometry concerns an old conjecture due to A. D. Aleksandrov, which states that if $S \subset \mathbb{R}^3$ is a closed convex surface of class C^2_+ such that

$$(k_1 - c)(k_2 - c) \leq 0,$$

with $c = \text{const}$, where k_1 and k_2 are the principal curvatures, then S must be a sphere of radius $1/c$.

It has an equivalent formulation in terms of hedgehogs. If $H_h \subset \mathbb{R}^3$ is a hedgehog such that $R_h \leq 0$, then H_h is a point. Martinez-Maure constructed a counterexample to the latter conjecture.

Matthieu Fradelizi spoke about Functional versions of L_p-affine surface area and entropy inequalities, which is a joint work with U. Caglar, O. Guéedon, J. Lehec, C. Schütt and E. Werner. Let us recall the definition of the L_p-affine surface area of a convex body. Let K be a convex body, containing the origin in its interior and $p \in \mathbb{R} \cup \{+\infty\}$, $p \neq -n$. Then the *L_p-affine surface area* of K is

$$as_p(K) = \int_{\partial K} \left(\frac{\kappa_K(x)}{\langle x, N_K(x)\rangle^{n+1}} \right)^{\frac{p}{n+p}} \langle x, N_K(x)\rangle dx,$$

where $N_K(x)$ is the unit normal to K at x and $\kappa_K(x)$ is the Gauss curvature.

This concept has been actively studied in recent years. In this work the authors introduce a functional version of the affine surface area. Let $\psi : \mathbb{R}^n \to$

$\mathbb{R} \cup \{+\infty\}$ be a convex function and $\lambda \in \mathbb{R}$. Define

$$as_\lambda(\psi) = \int_{\{\psi < +\infty\}} e^{(2\lambda-1)\psi(x) - \lambda\langle x, \nabla\psi(x)\rangle} \left(\det \nabla^2 \psi(x)\right)^\lambda dx.$$

It has many properties that hold for the usual affine surface area. Moreover, the latter can be obtained from the definition above by letting $\psi = \frac{1}{2}\|\cdot\|^2$ and $\lambda = \frac{p}{p+n}$.

There is also a duality relation similar to that for the usual L_p affine surface area, namely

$$as_\lambda(\psi) = as_{1-\lambda}(\psi^*),$$

where ψ^* is the Legendre transform of ψ.

They also established functional functional versions of affine isoperimetric inequalities.

Theorem 0.7 *If $\int xe^{-\psi(x)}dx = 0$ and $\lambda \in [0,1]$, then*

$$as_\lambda(\psi) \leq (2\pi)^{\lambda n} \left(\int e^{-\psi(x)} dx\right)^{1-2\lambda},$$

with equality if and only if $\psi(x) = \langle Ax, x\rangle + c$. The inequality is reversed if $\lambda < 0$.

The theorem, in particular, allows to recover the affine isoperimetry for convex bodies.

Fradelizi also discussed the following reverse log-Sobolev inequality. Let $\psi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a convex function such that $\int e^{-\psi(x)}dx = 1$. Then

$$S(\psi_0) - S(\psi) \geq \frac{1}{2} \int \log(\det \nabla^2 \psi) e^{-\psi} dx,$$

with equality if and only if $\psi(x) = |Ax|^2 + bx + c$. Here, $S(\psi)$ is the entropy of $e^{-\psi}$ and $S(\psi_0)$ is the entropy of the Gaussian.

The inequality is due to Artstein, Klartag, Schütt and Werner. In the present work the authors relaxed the smoothness hypotheses, established the equality case and gave a new proof.

A related theme was discussed in the talk by Elisabeth Werner on "Equality characterization and stability for entropy inequalities", joint work with T. Yolcu. Caglar and Werner recently proved the following result.

Theorem 0.8 *Let $f : (0,\infty) \to \mathbb{R}$ be a concave function. Let $\psi : \mathbb{R}^n \to R$ be a convex function that is in $C^1(\mathbb{R}^n)$. Then*

$$\int f\left(e^{2\psi - \langle\nabla\psi, x\rangle} \det(\nabla^2\psi)\right) d\mu < f\left(\frac{\int_{\mathbb{R}^n} e^{-\psi^*}dx}{\int d\mu}\right) \int d\mu. \qquad (1)$$

If f is convex, the inequality is reversed. Here, ψ^* is the Legendre transform of ψ.

Up to now there was no characterization of equality in the latter theorem. It can be shown that in the case of f being either strictly convex or strictly concave, equality holds in (1) if and only if ψ satisfies the Monge-Ampère equation

$$\det(\nabla^2\psi(x)) = \frac{\int e^{-\psi^*}dx}{\int e^{-\psi}dx} e^{-2\psi+\langle\nabla\psi(x),x\rangle}, \quad x \in \mathbb{R}^n. \tag{2}$$

Using methods from mass transport, due to Brenier and Gangbo-McCann, the authors show the uniqueness of the solution of (2).

Theorem 0.9 *Let f be a strictly convex, respectively concave, function. Then $\psi(x) = \frac{1}{2}\langle Ax, x\rangle + c$ is the unique solution of (2) and moreover equality holds in (1) if and only if $\psi(x) = \frac{1}{2}\langle Ax, x\rangle + c$. Here, c is a positive constant and A is an $n \times n$ positive definite matrix.*

The authors then give stability versions of (1), as well as for a reverse log Sobolev inequality and for the L_p-affine isoperimetric inequalities for both, log concave functions and convex bodies. In the case of convex bodies such stability results have only been known in all dimensions for $p = 1$ and for $p > 1$ only for 0-symmetric bodies in the plane.

Joseph Lehec gave a talk titled "Bounding the norm of a log-concave vector via thin-shell estimates", a joint work with Ronen Eldan. Chaining techniques show that if X is an isotropic log-concave random vector in R^n and Γ is a standard Gaussian vector then

$$\mathbb{E}\|X\| < Cn^{1/4}\mathbb{E}\|\Gamma\|$$

for any norm $\|\cdot\|$, where C is a universal constant.

Using a completely different argument the authors establish a similar inequality relying on the thin-shell constant

$$\sigma_n = \sup(\sqrt{var(|X|)} : X \text{ is isotropic and } \log - \text{concave on } \mathbb{R}^n).$$

In particular, they show that if the thin-shell conjecture $\sigma_n = O(1)$ holds, then $n^{1/4}$ can be replaced by $\log(n)$ in the inequality. As a consequence, they obtain certain bounds for the mean-width, the dual mean-width and the isotropic constant of an isotropic convex body. In particular, they give an alternative proof of the fact that a positive answer to the thin-shell conjecture implies a positive answer to the slicing problem, up to a logarithmic factor.

Ronen Eldan presented his work titled "A Two-Sided Estimate for the Gaussian Noise Stability Deficit". *The Gaussian noise-stability* of a set $A \subset \mathbb{R}^n$ is defined by

$$S_\rho(A) = \mathbb{P}(X \in A \;\&\; Y \in A)$$

where X, Y are standard jointly Gaussian vectors satisfying $\mathbb{E}[X_iY_j] = \delta_{ij}\rho$. Borells inequality states that for all $0 < \rho < 1$, among all sets $A \subset \mathbb{R}^n$ with a given Gaussian measure, the quantity $S_\rho(A)$ is maximized when A is a half-space.

Eldan gave a novel short proof of this fact, based on stochastic calculus. Moreover, he proved an almost tight, two-sided, dimension-free robustness estimate for this inequality: by introducing a new metric to measure the distance between the set A and its corresponding half-space H (namely the distance between the two centroids), he shows that the deficit $S_\rho(H) - S_\rho(A)$ can be controlled from both below and above by essentially the same function of the distance, up to logarithmic factors. As a consequence, he also established the conjectured exponent in the robustness estimate proven by Mossel-Neeman, which uses the total-variation distance as a metric. In the limit $\rho \to 1$, he obtained an improved dimension-free robustness bound for the Gaussian isoperimetric inequality. The estimates are also valid for a more general version of stability where more than two correlated vectors are considered.

The Ball as a Pessimal Shape for Packing was the topic of Yoav Kallus' talk. It was conjectured by Ulam that the ball has the lowest optimal packing fraction out of all convex, three-dimensional solids. A naive motivation is that the sphere is the most symmetric solid and therefore also the least free: in placing a sphere in space there are only three degrees of freedom, compared to five in the case of any other solid of revolution, and six in the case of any other solid.

On the plane the situation is different. It is known that the circle is not even a local pessimum (the opposite of an optimum). There are origin-symmetric shapes arbitrarily circular (the outradius and inradius both being arbitrarily close to 1) that cannot be packed as efficiently as circles. Ulam's conjecture implies that this would not be the case in three dimensions. The question then is in which dimensions, if in any, the ball is a local pessimum.

Here the author proved that any origin-symmetric convex solid of sufficiently small asphericity can be packed at a higher efficiency than balls. The author shows that the ball is not a local pessimum in dimensions 4, 5, 6, 7, 8, and 24. On the other hand, in the three-dimensional case the ball is proved to be a local pessimum.

Galyna Livshyts spoke about her work on "Maximal surface area of a convex set in $\mathbb{R}^n$ with respect to log-concave rotation invariant measures". Recall that the *Minkowski surface area* of a convex set Q with respect to the measure γ is defined to be

$$\gamma(\partial Q) = \liminf_{\epsilon \to 0^+} \frac{\gamma(Q + \epsilon B_2^n) \setminus Q}{\epsilon}$$

Let $\mathcal{K}^n$ be the set of all convex bodies in $\mathbb{R}^n$. It was shown by K. Ball that in the case when γ is the standard Gaussian measure, $\sup_{Q \in \mathcal{K}^n} \gamma(\partial Q)$ is finite and does not exceed $4n^{1/4}$. F. Nazarov showed that this order is actually the right one, and in fact,

$$0.28n^{1/4} < \sup_{Q \in \mathcal{K}^n} \gamma(\partial Q) < 0.64n^{1/4}.$$

The author establishes a similar result for all rotation invariant log-concave probability measures. It is shown that the maximal surface area with respect to such measures is of order

$$\frac{\sqrt{n}}{\sqrt[4]{Var|X|}\sqrt{\mathbb{E}|X|}},$$

where X is a random vector in $\mathbb{R}^n$ distributed with respect to the measure.

Peter Pivovarov spoke on "Volume of the polar of random sets and shadow systems", joint work with D. Cordero-Erausquin, M. Fradelizi , G. Paouris. The speaker discussed inequalities for the volume of the polar of random sets, generated for instance by the convex hull of independent random vectors in Euclidean space. Extremizers are given by random vectors uniformly distributed in Euclidean balls. This provides a randomized extension of the Blaschke-Santaló inequality. In particular, they prove the following.

Theorem 0.10 *Let N, $n \geq 1$. In the class of N-tuples $(X_1, ..., X_N)$ of independent random vectors in $\mathbb{R}^n$ whose laws have a density bounded by one, the expectation of the volume of the set*

$$(\mathrm{conv}\{\pm X_1, ..., \pm X_N\})^\circ$$

is maximized by N independent random vectors uniformly distributed in the Euclidean ball $D_n \subset \mathbb{R}^n$ of volume one.

This theorem, in particular, implies the classical Blaschke-Santaló inequality for convex bodies.

Luis Rademacher gave a talk titled "The More, the Merrier: the Blessing of Dimensionality for Learning Large Gaussian Mixtures", based on joint work with J. Anderson, M. Belkin, N. Goyal and J. Voss. He discussed recent developments in high dimensional geometric statistical inference. The problems included the reconstruction of polytopes from uniformly random points and the estimation of parameters of Gaussian Mixture Models. The main questions were the stability of the estimation and the design of efficient algorithms. The techniques involved provably correct and stable versions of the method of moments and more general harmonic analysis. The main original result is a somewhat unexpected "blessing of dimensionality", where it was shown that the problem of estimating the parameters of a Gaussian Mixture Model is generically easy in high dimension, while it is generically hard in low dimension.

Konstantin Tikhomirov spoke on the distance of polytopes with few vertices to the Euclidean ball. Let n, N be natural numbers satisfying $n+1 \leq N \leq 2n$, B_2^n be the unit Euclidean ball in $\mathbb{R}^n$ and $P \subset B_2^n$ be a convex n-dimensional polytope with N vertices and the origin in its interior. He proved that

$$\inf\{\lambda \geq 1 : B_2^n \subset \lambda P\} \geq cn/\sqrt{N-n},$$

where $c > 0$ is a universal constant (this solves a conjecture of Gluskin and Litvak). As an immediate corollary, for any covering of S^{n-1} by N spherical caps of geodesic radius ϕ, $\cos(\phi) \leq C\sqrt{N-n}/n$ for an absolute constant $C > 0$. Both estimates are optimal up to the constant multiples c, C.

Pierre Youssef spoke about "Almost orthogonal contact points". Let $K \subset \mathbb{R}^n$ be a convex body in John's position. The aim is to find a "large" number of contact points which is almost equivalent to an orthonormal basis of l_2.

Recall that a sequence of unit vectors $\{y_j\}_{j\le k} \in \mathbb{R}^n$ is said to be C-equivalent to the canonical basis of ℓ_2^k if there exists α, $\beta > 0$ such that $C = \beta/\alpha$ and for any scalars $\{a_j\}_{j\le k}$

$$\alpha \left(\sum_{j\le k} a_j^2 \right)^{1/2} \le \left\| \sum_{j\le k} a_j y_j \right\|_2 \le \beta \left(\sum_{j\le k} a_j^2 \right)^{1/2} .$$

The author proves the following.

Theorem 0.11 *Let $\epsilon \in (0,1)$, there exists $r = r(\epsilon)$ such that the following is true. Let $n \in \mathbb{N}$ and K a convex body in $\mathbb{R}^n$ such that B_2^n is the ellipsoid of maximal volume contained in K. There exists a basis of $\mathbb{R}^n$ formed by contact points x_1,..., x_N and there is a partition of $[n]$ in r sets σ_1,..., σ_r such that $\forall i \le r$, $\{x_j\}_{j\in\sigma_j}$ is $(1+\epsilon)$-equivalent to the canonical basis of $\ell_2^{\sigma_i}$.*

This is proved as a consequence of a more general result which asks about extracting a well conditioned block inside a given matrix.

Carsten Schütt gave a talk on "Order statistics" (joint work with S. Kwapien). The expression $\underset{1\le i\le n}{\text{k-min}}\, a_i$ denotes the k-smallest of all the numbers $a_1, \ldots, a_n$.

Let $M = (M_1, \ldots, M_n)$ be an n-tuple of increasing maps from $[0, \infty)$ onto itself. Then for all $x \in \mathbb{R}^n$ we define

$$\|x\|_M = \inf \left\{ \rho \left| \sum_{i=1}^{n} M_i \left(\frac{x_i}{\rho} \right) \le 1 \right. \right\} .$$

Theorem 0.12 *There is a universal constant C such that for all independent, positive and uniformly subregular sequences of random variables $\xi_1, \ldots, \xi_n$ with continuous distribution functions $F_1, \ldots, F_n$*

$$\mathbb{E} \left(\underset{1\le i\le n}{\text{k-min}}\, \xi_i \right) \sim \text{med} \left(\underset{1\le i\le n}{\text{k-min}}\, \xi_i \right) \sim \frac{1}{\|(1, \ldots, 1)\|_{\frac{F}{k-\frac{1}{2}}}} .$$

Petros Valettas gave a talk on "Neighborhoods on the Grasmannian of marginals with bounded isotropic constant", joint work with G. Paouris.

Recall that the isotropic constant of a centered log-concave probability measure on $\mathbb{R}^n$ is defined as

$$L_\mu = \|\mu\|_\infty^{1/n} \left[\det \left(\int_{\mathbb{R}^n} x_i x_j d\mu(x) \right) \right]^{1/(2n)} ,$$

where $\|\mu\|_\infty = \|f_\mu\|_\infty$ and f_μ is the density function of μ.

If μ is isotropic, then $L_\mu = \|\mu\|_\infty^{1/n}$. The hyperplane conjecture can be formulated in the form: there exists a constant $C > 0$ such that for all n and any μ log-concave isotropic probability measure on $\mathbb{R}^n$, we have $L_\mu < C$.

The authors proved a theorem in the spirit of Klartag's isomorphic version of the hyperplane conjecture. Consider the following distance on the Grassmanian. For $E,\, F \in Gr(n,k)$, let

$$d(E,F) = \inf\{\|I-U\|_{op} : U \in O(n),\ U(E)=F\}.$$

The main result is as follows.

Theorem 0.13 *Let μ be an isotropic log-concave probability measure on $\mathbb{R}^n$ and let $1 \le k \le \sqrt{n}$. For every $\epsilon > 0$ and any $E \in Gr(n,k)$ there exists $F \in Gr(n,k)$ with $d(E,F) < \epsilon$ such that*

$$L_{\pi_F \mu} \le C/\epsilon,$$

where $C > 0$ is an absolute constant. Additionally, if L_μ is bounded, then one can take $1 \le k \le n-1$.

Patrick Spencer presented "A note on intersection bodies and Lorentz balls in dimensions greater than 4".

Let K be an origin-symmetric star body in $\mathbb{R}^n$. A star body L is called the intersection body of K if

$$\rho_L(\xi) = |K \cap \xi^\perp|, \quad \xi \in S^{n-1},$$

where $\rho_L(\xi)$ is the radius of L in the direction of ξ and $\xi^\perp$ is the hyperplane through the origin perpendicular to ξ.

The closure (in the radial metric) of the class of intersection bodies of star bodies is called the class of intersection bodies.

Let $w = (a_1, ..., a_n) \in \mathbb{R}^n$ be such taht $a_1 \ge a_2 \ge \cdots \ge a_n \ge 0$. Define a norm on $\mathbb{R}^n$ as

$$\|x\|_{w,q} = \left(a_1 (x_1^*)^q + \cdots a_n (x_n^*)^q\right)^{1/q},$$

where x_1^*,..., x_n^* is the non-increasing permutation of $|x_1|$,..., $|x_n|$.

The space $\ell_{w,q}^n = (\mathbb{R}^n, \|\cdot\|_{w,q})$ is called the Lorentz space. The following result is proved.

Theorem 0.14 *Let $n \ge 5$ and $q > 2$. The function $\|x\|_{w,q}^{-p}$ represents a positive definite distribution if and only if $p \in [n-3, n)$.*

In particular, the unit ball of $\ell_{w,q}^n$ is not an intersection body for $q > 2$ and $n \ge 5$.