Rigorously Verified Computing for Infinite Dimensional Nonlinear Dynamics (14w5098)

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1 Overview of the Field

Partial differential equations are at the core of the mathematical description of the world around us: from Schrödinger's equation in quantum mechanics to the Navier-Stokes equations in fluid dynamics, from the reaction-diffusion equations governing biochemistry to the Black-Scholes equation in mathematical finance.

The nonlinear nature of many of these equations makes their analysis challenging. Two complementary approaches are used to investigate such problems. On the one hand, one can exploit geometric and topological ideas to perform a global analysis. This provides robust qualitative information. Ideally this is coupled with numerical calculations, which offer detailed quantitative information and clear pictures of the solutions, but the information from the numerics is local (in parameter space) and non-rigorous.

Dramatic advances in algorithms, analysis, code, and computer speed and memory have opened the possibility of utilizing the power and robustness of topological and analytic methods to rigorously verify computational results. For questions related to nonlinear dynamics the most significant results, e.g. [1, 2], are associated with finite dimensional systems. Computer assisted proofs of the existence of low dimensional dynamical structures, e.g. fixed points, periodic orbits, heteroclinic and homoclinic orbits, can be used as building blocks in global analysis, either using gluing methods from dynamical systems theory or via Morse-Conley-Floer theory. In this way local, rigorously verified, numerical solutions form the seeds of information from which additional global understanding can be gained.

Moreover, encouraging first steps for infinite dimensional systems are starting to appear. In this workshop we explored the challenges that lie ahead in applying these techniques to fully fledged problems in the theory of infinite dimensional nonlinear dynamical systems, with a particular emphasis on nonlinear partial differential equations.

Rigorous verification goes far beyond an a posteriori analysis of numerical computations. In a nutshell, verification methods are mathematical theorems formulated in such a way that the assumptions can be rigorously verified on a computer. Indeed, it requires an a priori setup that allows analysis and numerics to go hand in hand: the choice of function spaces, the choice of the basis functions/elements and Galerkin projections, the analytic estimates, and the computational parameters must all work together to bound the errors due to approximation, rounding and truncation sufficiently tightly for the verification proof to go through. On top of that, for high and infinite dimensional problems additional aspects arise. On the analysis side, we need to deal with much subtler and more involved truncation estimates and, for connecting orbits, with high or infinite dimensional invariant manifolds. On the algorithm side, we must find suitable pre-conditioners and develop efficient interval-arithmetic based algorithms. Finally we need to understand how to tie these computational results to the geometric and topological ideas of global nonlinear analysis that form the framework for our understanding of nonlinear dynamics.

2 Presentations

We had presentations at the start of each morning and afternoon session (except for Wednesday afternoon, when we hiked up Sulphur Mountain).

1. Jean-Philippe Lessard, Rigorously verified computing for infinite dimensional nonlinear dynamics: a functional analytic approach

2. Siegfried Rump, Computer-assisted proofs using floating point arithmetic using the new INTLAB

3. Jay Mireles-James, Fixed point approach to rigorous validated computation of connecting orbits in infinite dimensions

4. Piotr Zgliczynski, Geometric methods in the integration of evolutionary problems in infinite dimension

5. Michael Plum, Computer-assisted existence and multiplicity proofs for semilinear elliptic boundary value problems

6. Christian Reinhardt, Rigorous numerics using Chebyshev series
   and

Jacek Cyranka, Some results on global attractors of certain parabolic PDEs and a 2D convection-diffusion PDE

7. Arnold Neumaier, Rigorously covering all solutions of infinite-dimensional equations

Additionally, on Monday the PhD students introduced themselves in a sequence of short talks:

1. Ray Sheombarsing, Rigorous numerical methods for dynamical systems
2. Andréa Deschênes, Coexistence of hexagons and rolls
3. Jonathan Jaquette, Parametrizing Invariant Manifolds for Flows in Banach Spaces
4. Chris Groothedde, Rigorous numerics
5. Maxime Murray, The suspension bridge equation
6. Aleksander Czechowski, Periodic orbits of the FitzHugh-Nagumo equations — a computer assisted proof
7. Maxime Breden, Rigorous numerics for a tridiagonal dominant operator

3 Outcome of the Meeting

The main goal when structuring this workshop was to create plenty of opportunities for interaction and collaboration. After each lecture we had a central meeting to make a list of possible subjects to work on, such as an open problem or a particular technique (adding and removing subjects to/from the list as the week went on). We then democratically selected three topics for discussion. The remainder of the morning or afternoon was spent in breakout sessions: smaller subgroups came together to exchange ideas. Some topics were discussed just once, others multiple times. The results of the nine selected topics are summarized below.

3.1 Software for rigorous verification of PDEs

The functional analytic approach to rigorous verification of PDEs is still a new field but has progressed theoretically, and gained sufficient researchers. While scientific software is a necessary product of research in rigorously verified computing, a conscious effort is needed to collect and share the disparately developed software. By having a community curated collection of code, we hope to encourage researchers to improve their coding practices and allow for a greater degree of software-reusability.

We have some preliminary ideas for good practice for computer assisted proofs in this area such as:

- Readable, commented code
- Documentation to accompany publicly available code
- Variable names and algorithmic implementations in the code should match those the accompanying paper
- Scripts reproducing any and all figures in the accompanying paper should also be publicly available.

All aspects of a proof must be clear to a reader of both the paper and public code. All results should be completely reproducible!

There is no reason to try and enforce a standard language or coding style on the community. Instead we intend to create a wiki to share work in a common place that is open to all and editable by all. By developing a wiki-based repository maintained by the community, its continued existence will not be dependent on any one person or group. We hope that good practice will develop from researchers have good examples to follow and start from.

Initially the wiki will primarily be a repository for papers, codes and links. We hope that over time we can include an index of methods, subroutines, problems, etc., so that people can quickly find what they are looking for.

It is also important to start looking at the functional analytic approaches in comparison to the more geometric approach. We need to compare problems with those that can be done with CAPD to find where which approach is better and where the two methods are complimentary. Additionally we plan to collect test examples such as descriptions and implementations of simple rigorous ODE integrators, examples of continuation proofs and other standard infinite-dimensional problems. These examples can then be used by newcomers, allowing for a more open and inviting introduction to the field of rigorously verified computations than previously possible.

Finally, we identified several directions for common software needs, such as a rigorous FFT algorithm and rigorous continuation wrappers.

3.2 Fast Fourier Transform

For a number of the problems discussed in our workshop the practical computation of rigorous error bounds uses the Fast Fourier Transform (FFT). As the name suggests it is a fast version of the Discrete Fourier Transform (DFT) for converting time (or space) data into a frequency domain and vice versa. There are a number of algorithms for converting an n-vector by FFT, and those algorithms are especially efficient if n is a power of 2.

On a broader scale the computation of rigorous error bounds requires, in particular, the estimation of rounding errors. Rather than doing that for each operation individually it is very useful not to use plain floating-point arithmetic but to handle the estimates by special data types, for example, interval arithmetic.

To address operations on such new data types using traditional programming language C or Fortran and corresponding software is quite cumbersome. This can be solved by using Matlab. This very widely used programming environment offers an operator concept. That in turn allows to write operations including rigorous error estimates at a level close to mathematical notation. This is true for scalars, vectors, matrices and other spaces.

A convenient Matlab toolbox to address required operations with rigorous error bounds is INTLAB, the Matlab toolbox for reliable computing. A number of participants in the workshop use INTLAB. For some reason the Fast Fourier Transform is not (yet) included in INTLAB. It turned out during the workshop that first, FFT with rigorous error bounds is very important and second, there is an existing routine which works correctly but, however, is rather slow.

One main reason for weak performance is the fact that Matlab interprets the code. On the one hand this allows to write a code close to the mathematical specification. However, on the other hand, interpretation may slow down computations significantly. This can be avoided by vectorizing Matlab code. In that case the computing time of a single operation can be much larger than the interpretation time of that single operation, thus diminishing the interpretation overhead.

The existing routine for the FFT was vectorized to a certain degree, but more could be done. As a result of the workshop a new routine verifyfft for the forward and inverse FFT with rigorous error bounds was developed and distributed among the participants. For small vector lengths the computing was reduced by a factor of more than 100, for larger vector lengths even by a factor 1000 and more.

Moreover, often many vectors have to be transformed at a time. In that case it is superior to put the vectors into a matrix and to transform all columns of the matrix at once. This kind of vectorization improves the performance again by an order of magnitude for moderate vector lengths.

Both the existing and the new routine compute verified inclusions of the true result. The results of the existing routine are pretty accurate, i.e. the bounds are narrow. An additional effort was spend to improve the width of the bounds. For moderate vector lengths the improvement is about two orders of magnitude.

The workshop helped to identify the FFT with verified error bounds as a crucial operation in many of the algorithms for computing rigorous error bounds for continuous problems. The new routine will be included in INTLAB.

3.3 Nonexistence Proofs for Dynamical Systems

This discussion focused on the question of whether one can compute the entire solution set for problems of the type $F(x) = 0$, $x \in X$ where X is a Banach space. This involves not only locating and verifying solutions, but also ruling out solutions in the remainder of X. Ruling out additional solutions, or showing the

nonexistence of solutions, requires a number of reductions for infinite-dimensional problems. This may include using analytical methods to determine feasible bounds of the solution set in X and using computational methods to refine this set by locating regions on which one can prove that no solution exists.

We discussed two prior results to motivate our more general discussion. Day and Kalies (2013) used two sets of bounds (labeled *global* and *point-wise*) together with computational outer approximation techniques to bound and refine a covering of the maximal invariant set for a class of integrodifference equations with smooth nonlinearities. Cyranka and Zgliczyński (arXiv:1403.7170, 2014) used analytical bounds and computations of absorbing sets to prove that a unique equilibrium solution is globally asymptotically stable (hence, ruling out additional solutions) for the viscous Burgers equation.

One of the outcomes of the further discussions was the recognition that it is possible to reduce (under favorable conditions) the problem of a rigorous enclosure of all solutions of equations in a Banach space with reasonable tolerances to a finite-dimensional constraint satisfaction problem (CSP) whose solution gives the desired information about the original problem. If a low number of degrees of freedom suffice to give a reasonable approximate description of the problem, if the number of solutions is not too large, and if these solutions are not overly sensitive to perturbations of the problem then the CSP is expected to be solvable in a reasonable amount of time by off-the-shelf rigorous branch-and-bound methods for constraint satisfaction problems. The resulting rigorous error bounds for the solutions of the infinite-dimensional problem are then expected to be reasonable. This resulted in a lecture by Arnold Neumaier on Thursday afternoon, and a manuscript written during the meeting.

Several software packages exist for solving factorable CSPs with complete rigor: COCONUT, GloptLab, ANTIGONE, Couenne, ICOS, and GlobSol. In each case, the user only needs to provide an appropriate text file describing the constraints and to select appropriate options for steering and stopping the solver. The prize-winning solver BARON also solves constraint satisfaction problems, but relies on unverified techniques that may lead to inaccurate coverings and even to the loss of solutions.

As one possible test case the group briefly discussed the diblock copolymer model, which is one of the standard equations describing micro-phase separation in materials. It is a fourth-order parabolic partial differential equation, which arises as a regular perturbation of the well-studied Cahn-Hilliard model. The model is a gradient system, so the study of its global dynamics reduces to understanding its global attractor. It was shown in Johnson, Sander, and Wanner (2013) that the diblock copolymer bifurcation diagram is considerably more complicated than that of the Cahn-Hilliard model. At the same time, low-dimensional projections of the flow seem to provide reasonable approximations of the dynamics in interesting parameter regimes. As a test case, the group created GAMS input files for a three-dimensional projection of the model, and used BARON through the NEOS server to perform a comprehensive sweep of the equilibrium structure at a particular parameter value. BARON resolved all fifteen equilibrium solutions of the model in a few minutes, in accordance with the results of Johnson, Sander, and Wanner (2013).

3.4 Approximate inverse of infinite dimensional tridiagonal operators

Consider a tridiagonal linear operator of the form

$$J = \begin{pmatrix} D & & & 0 & & \\ & \lambda & & & & \\ & \lambda & \mu_m & \lambda & & \\ 0 & & \lambda & \mu_{m+1} & \lambda & \\ & & & \ddots & \ddots & \ddots \end{pmatrix}, \tag{1}$$

where D is a finite dimensional $m \times m$ matrix, λ is a large constant and $|\mu_k| \to \infty$ as $k \to \infty$. An operator of the form (1) occurs in several contexts, and obtaining explicit bounds on its inverse, or constructing a good approximate inverse is a key step in many rigorous computational methods aimed at studying infinite dimensional nonlinear problems. We considered two separate situations.

3.4.1 Non uniqueness phenomena in a convection-diffusion 2D PDE model

Consider the following convection-diffusion 2D PDE model

$$P\boldsymbol{u} \cdot \nabla \boldsymbol{u} - \Delta \boldsymbol{u} = \lambda \boldsymbol{F}(x), \qquad \boldsymbol{u} \colon \mathbb{T}^2 \to \mathbb{R}^2,\ \boldsymbol{F} \colon \mathbb{T}^2 \to \mathbb{R}^2, \tag{2a}$$

where $P\boldsymbol{u}$ is the Helmholtz projector onto the zero divergence subspace, i.e. $P\boldsymbol{u} = \boldsymbol{u} - \nabla \mathrm{div}\left(\Delta^{-1}\boldsymbol{u}\right)$, and $\mathbb{T}^2$ is the two dimensional torus. This is a toy model related with 2D incompressible Navier-Stokes equations. Our goal in studying this problem is to explore the non uniqueness phenomena – generically embedded in fluid dynamics models. In particular, we aim at answering the question of multiplicity of solutions with respect to the parameter λ. For the particular choice of forcing

$$F(x) = (2\sin x_2, 2\sin x_1),$$

when $\lambda \gg 1$, the stable solution can be decomposed into the dominant part $(2\lambda \sin x_2, 0)$, and the remainder $\widetilde{u}_\lambda$, i.e.

$$u_\lambda = (2\lambda \sin x_2, 0) + \widetilde{u}_\lambda. \tag{3}$$

After this formula is plugged into (2), the linear part takes the form of the operator

$$L(\widetilde{u}) = 2\lambda \sin x \widetilde{u}^2 + \Delta \widetilde{u}^2,$$

which is of tridiagonal form as in (1) with $\mu_k = -1 - (k-1)^2$ in Fourier's basis.

3.4.2 Results and conjectures

The problem of bounding $\|L_\lambda^{-1}\|$ (in an operator norm) is crucial for establishing the existence of u_λ – solution of (2) asymptotically for large λ values, which we conjecture is true. Our current research in progress revealed that if we consider a finite k-dim truncation of the operator L_λ ($P_k L_\lambda$) its norm can be bounded

$$\|(P_k L_\lambda)^{-1}\|_\infty < C/\sqrt{\lambda},$$

where the constant C does not depend on both of the truncation dim. k and the parameter λ. On the workshop we realized that this bound probably holds even for the inverse of the infinite dim operator L_λ, and this could help us progress with our conjecture.

3.4.3 Chebyshev series for the study of BVP

When using Chebyshev series to compute solutions to boundary value problems (BVP), we derive an infinite dimensional nonlinear operator whose derivative is of the form (1), where the off diagonal term λ consists essentially of a time rescaling constant multiplied by the first Chebyshev coefficients of the numerical approximation. Hence, for BVP arising in fast-slow systems or posed on long time domains, the off diagonal term will be large and hence should not be ignored when constructing an approximate inverse. During the week, we discussed about possibilities to adapt the work of [Breden, Desvillettes and Lessard, *Rigorous numerics for nonlinear operators with tridiagonal dominant linear parts*, preprint, 2014] to this context. Since there, the operators considered have unbounded off diagonal terms, we believe that the method should in principle be adapted.

3.5 Bifurcations

We discussed the problem of finding bifurcation points in infinite-dimensional parameter-dependent zero-finding problems

$$F(b, \lambda) = 0,$$

for $F : B \times \Lambda \to B$.

Among other things, we designed a general setup in the spirit Lyapunov-Schmidt reduction and considered variations thereon. Furthermore, we discussed the inherent problems surrounding of finding bifurcation-points in an asymmetrical setting using validated numerical techniques, i.e. singularities of the derivative, pitchfork, period-doubling etc. Finally, we considered several examples, such as the pitchfork's normal form, the Swift-Hohenberg equation and a system of reaction-diffusion equations considered by J.-P. Lessard, and M. Breden.

3.5.1 Lyapunov-Schmidt-type method

More precisely, we considered a singular zero-finding problem $F(b_0, \lambda_0) = 0$ with a one-dimensional kernel

$$N = \ker(D_b F(b_0, \lambda_0)),$$

and one-dimensional co-range Y, whose complement is given by

$$R = \operatorname{ran}(D_b F(b_0, \lambda_0)).$$

Using the splittings $B = N \oplus X$ and $B = R \oplus Y$ one defines the one-dimensional projections $P : B \to N$ and $Q : B \to Y$. Consequently, the system $F(b, \lambda) = 0$ can be written equivalently as

$$\begin{aligned} QF(P(b) + (I - P)(b), \lambda) &= 0 \\ (I - Q)F(P(b) + (I - P)(b), \lambda) &= 0. \end{aligned}$$

The benefit of this construction is that the function $\Phi : N \times X \times \Lambda$, given by

$$\Phi(n, x, \lambda) = (I - Q)F(n + x, \lambda)$$

has a non-singular derivative (with respect to x) at n_0, x_0, λ_0 given by

$$D_x \Phi(n_0, x_0, \lambda_0) = (I - Q)D_x F(n_0 + x_0, \lambda_0).$$

By the implicit function theorem, this implies that there exists a function ϕ defined on a neighbourhood of $(n_0, \lambda_0) \in N \times \Lambda$, such that

$$\Phi(n, \phi(n, x), \lambda) = 0,$$

for all (n, λ) in this neighbourhood.

Using this, we can define the bifurcation function

$$g(n, \lambda) = QF(n + \phi(n, \lambda), \lambda),$$

whose derivatives can be used to determine the characteristic behaviour of the bifurcation. Crucial in this discussion is the verified computation of the function ϕ, which can be done for instance in the setting of radii-polynomials.

An alternative approach was considered based on regularizing the problem in the following way. Instead of splitting the space, one can augment the system by considering the problem given by

$$\begin{aligned} F(b, \lambda) - \kappa y &= 0 \\ \psi(b) - \mu &= 0. \end{aligned}$$

In this system y must be taken from the co-range of $DF(b_0, \lambda_0)$ and ψ is a suitably chosen monitor function, typically the function used to plot the bifurcation diagram.

This new system is non-singular and by solving it we can rigorously compute expressions for

$$\begin{aligned} b &= b(\kappa, y) \\ \kappa &= \kappa(\lambda, \mu). \end{aligned}$$

By studying the null set of κ, one can identify a neighbourhood of the bifurcation diagram and by subsequently considering the sign changes of κ one can verify the existence of a bifurcation branch contained in this neighbourhood. It should be noted that this technique usually is unable to rigorously verify the existence of a bifurcation point in cases where an inherent symmetry of the system cannot be exploited.

3.5.2 Asymmetrical systems

In addition, we also discussed the identification of bifurcations in settings where symmetry is either absent or not immediately obvious. Primarily, we considered this problem in the context of the pitchfork bifurcation. When such a bifurcation occurs, a symmetry in the system (usually given by the fact that F is an odd function) can be exploited in order to rigorously construct the bifurcation branch by dividing out the singularity of the system. This can then be used to prove the existence of a bifurcation point on this branch.

In cases where symmetry is not present, this cannot be done. In generic bifurcations, it is usually impossible to distinguish between two non-intersecting bifurcation branches and an actual pitchfork. We briefly discussed a method of finding bifurcation points by considering small perturbations of the original system, by adding a parameter to the system.

3.5.3 Examples

In conclusion, we considered several examples of systems in which bifurcations play a role. In particular, we briefly considered the pitchfork bifurcation's normal form while studying the problem of asymmetrical system.

In addition we also considered how our Lyapunov-Schmidt type method could be applied to the Swift-Hohenberg equation as well as a system of reaction-diffusion equations previously considered by J.-P. Lessard, and M. Breden.

3.6 Exploiting smoothing property of parabolic PDEs in rigorous numerics

Consider the problem

$$u_t(t,x) = Lu + N(u, Du, \dots, D^r u), \tag{4}$$

on some compact domain $\Omega \subset \mathbb{R}^n$ with some boundary conditions, where L - smoothing operator, Laplacian or its power with a correct sign and $r < s$, where s the order of L.

The smoothing properties of the time evolution of (4) are well known. When the periodic boundary conditions for (4) are considered, then in the Fourier basis $\{\exp(ikx)\}_{k\in\mathbb{Z}^n}$ the effect of smoothing gives rise for tail isolation for the sets of the form $W \oplus \Pi_{|k|>M} B(0, \frac{C}{|k|^s})$. This makes it possible to design a rigorous integrator for such system.

Two crucial facts in the construction of the integrator are

- L is diagonal in the Fourier basis, hence $\exp^{Lt}$ is known explicitly;
- the following lemma holds:

  Lemma 1. *Let $s > s_0$. If $|a_k| \le C/|k|^s$, $|a_0| \le C$, then there exists $D = D(C,s)$*

  $$|N_k| \le \frac{D}{|k|^{s-r}}, \qquad |N_0| \le D.$$

  This is in fact a statement about regularity, namely if a is of "class C^s" (plus bounds) then $N(a)$ is of "class C^{s-r}" (plus bounds).

For other boundary conditions and domains usually we do not have the eigenfunctions basis and Lemma 1 appears not to be true. This probably excludes the possibility of the component-wise isolation with arbitrary decay power s. However it is quite possible that we will have an isolation, if in the tail we consider balls in some norms. Instead of (4) we think it is worth trying to analyze its integral form

$$u(t) = \exp(Lt)u(0) + \int_0^t \exp(L(t-t'))N(u(t'), Du(t'), \dots)dt' \tag{5}$$

Observe that $\exp(Lt)$ is the Green function for the problem (4) with $N \equiv 0$.

We can rewrite (5) as follows

$$u(t,x) = \int_\Omega G(t,x')u_0(x')dx' + \int_0^t dt' \int_\Omega dx' G(t-t', x-x')N(u(t',x')) \tag{6}$$

We see two possibilities of splitting of our phasespace, $X \oplus Y$, into 'main modes' X and the tail Y

- using approximate leading eigenfunctions for L generating X and estimating action of L on $Y = X^{\perp}$. The techniques used by Plum and his coworkers can be helpful in obtaining this goal. On the other side it not obvious how to exploit the smoothing property.

- assuming some knowledge of G (some bounds which allow to derive the smoothing property) following the approach developed by Nakao we set $X = S_h$ and $Y = S_h^{\perp}$ where S_h is obtained as the projection trough some interpolation. Hopefully in this setting the isolation property can be analytically established. This requires study of two topic: developing bounds for the projection error in various norms and function spaces and the Green function for the linear problem.

We think that it will be worth to look at a problem on the disk, where the eigenfunctions are known (Bessel functions) but apparently Lemma 1 is not true. There we should learn how to achieve an isolation and this new insight will likely allow us to attack other problems.

3.7 Eigenvalue enclosures and exclosures

In various contexts of rigorous computations, but also of more general analytical problems, information about the spectrum of given selfadjoint or non-selfadjoint eigenvalue problems is of crucial importance. For example, when the stable and the unstable manifold of a dynamical system shall be suitably controlled by rigorous computations, information about the location of eigenvalues in certain parts of the complex plane is required. Also in norm computations for linear operators eigenvalue bounds play an important role.

For selfadjoint eigenvalue problems, eigenvalues below the essential spectrum can be characterized variationally e.g. by Poincare's min-max-principle. Based on such characterizations, finite dimensional matrix eigenvalue problems can be put up, the eigenvalues of which are upper or lower bounds for the eigenvalues of the given (usually infinite dimensional) eigenvalue problem. These matrix eigenvalue problems can be solved rigorously by means of well-established linear algebra software, as e.g. INTLAB. The eigenvalue bounds obtained in this way are index-wise, and thus in particular also guarantee that the regions (on the real line) between the enclosing intervals are free of eigenvalues, i.e. they also give eigenvalue exclosures.

For non-selfadjoint eigenvalue problems, the situation is more complicated since no useful variational eigenvalue characterizations are available. Here, after a suitable reformulation of the problem, fixed-point arguments can be used to rigorously enclose single eigenvalues in the complex plane, and also for computing compact complex regions which are free of eigenvalues, i.e. eigenvalue exclosures.

For the more special case of linear operators acting in spaces of (real or complex) sequences, which therefore are directly representable as infinite dimensional matrices, also Gerschgorin-type eigenvalue enclosures can be considered. In particular when the off-diagonal entries are "small" (measured in the corresponding sequence-space norm), such bounds are often quite satisfactory. For example, in case of the Kot-Schafer model such Gerschgorin-type bounds are sufficient to obtain the desired information about the number of eigenvalues located inside or outside the complex unit disk, as needed for controlling the stable and the unstable manifold.

It will be of major interest for future investigations to identify larger classes of operators, also out side the class of operators in sequence spaces, for which Gerschgorin-type eigenvalue bounds give appropriate information.

3.8 Perturbation Arguments in the Method of Radii Polynomials

Many problems in analysis are solved by formulating a zero finding problem for a function between appropriate Banach spaces. Indeed this was a recurring theme in a number of the sessions and many of the lectures at this workshop. A number of researchers at the workshop have used the method of radii-polynomials in order to organize computer assisted zero finding arguments. This method begins with a particular numerical approximate solution, a particular choice of approximate inverse for the problem, and a choice of the Banach space on which the solution is desired. One constructs a "Newton-like" operator and aims to shows that the operator is a contraction in some neighborhood of the approximate solution.

Based on these choices the method of radii polynomials is a strategy for obtaining bounds on the smallest and largest neighborhoods about the approximate solution on which the Newton-like operator is a contraction

mapping. The radii of these balls appear as the roots of some polynomial equations with coefficients determined by the data of the problem. The size of the smallest of the neighborhoods gives a rigorous bound on the truncation error. The size of the largest neighborhood gives isolation bounds for the solution. Continuity of the radii polynomials aids in the smooth connection one computer assisted proof to another. In addition, information about the isolation of the solution is sometimes exploited to aid in mathematically rigorous continuation arguments.

This session considered the possibility of applying the radii-polynomial argument directly to the projected finite dimensional problem and then using some weaker perturbative argument to obtain a (possibly non-unique) solution of the infinite dimensional problem. We discussed the possibility of using the isolation information from the finite dimensional radii polynomial argument in order to facilitate the perturbative step. The participants observed that when the Newton like operator has certain smoothing properties then the perturbative argument seems to goes through. The argument is based on the Schauder theorem and requires some bounds on the difference between the projected and infinite dimensional map. The advantage of this approach is that projection errors for the derivative of the infinite dimensional map are not needed. The disadvantage is of course the loss of uniqueness.

Many problems considered in applied mathematics have the smoothing properties required for this argument. The participants also discussed the possibility that in some problems it may be possible to recover the uniqueness by some regularity/bootstrap arguments. Further consideration is required in order to determine if the proposed approach will lead to solution of problems not solvable by current methods. It does seem clear that the argument could be used in some cases to simplify some computer assisted proofs.

3.9 Fast-Slow systems

Fast-slow systems are difficult to analyse using rigorous numerics due to the presence of different time scales. In the singular limit, complementary topological-based techniques can be used to describe the dynamics for all $\epsilon \in (0, \epsilon_0]$, for some (very) small explicit ϵ_0. The problem to be solved is then to develop a rigorous numerical procedure for continuing the solution starting at $\epsilon = \epsilon_0$. In this regime the topological technique no longer works. Moreover, since ϵ_0 is (very) small, the time scale separation still hampers general continuation techniques. Hence, an approach adapted to fast-slow systems is needed.

We identified a hierarchy of problems:

1. *0th order*: the limit where a periodic orbit tends to a homoclinic one, so that part of the trajectory is close to an equilibrium (a 0-dimensional slow manifold).

2. *1st order*: trajectories that alternately follow one-dimensional slow manifolds and fast transitions between these.

3. *2nd order*: the case in which the slow manifold is of dimension two or higher, so that part of the problem is to determine *which* orbit on the slow manifold is being shadowed.

Concerning the 0th order problem, we focused on the functional analytic approach (in particular, using domain decomposition and Chebyshev series) for the Lorenz system. Several ideas were mentioned.

- Reparameterize time for the part of the trajectory that is close to the equilibrium and find a way to do this automatically (to be able to do continuation).

- Use splines near the equilibrium, as different bases functions can be used on each domain.

- Use normal forms (a la Tucker) to give a phase-space description near the equilibrium and combine that with a boundary value problem, which in turn can be solved by using the Chebyshev series.

- Combine the idea of normal forms with the choice of appropriate basis functions for the part of the trajectory near the equilibrium, i.e., try to lift the normal approach to function space.

Naturally, the 0th order problem has been solved before, using several methods (including CAPD). The goal of looking for new approaches is to use these as stepping stones for the 1st (and later 2nd) order problem. Concerning the latter, the ideas were obviously less concrete:

- Explicitly split the problem into several boundary values problems, each dealing with either a fast part or a slow part, so that time can be appropriately rescaled on each piece.
- Construct a parameterized normal form near the slow manifold. Here one could either use the slow manifold of the singular limit or a persisting slow manifold for small $\epsilon > 0$.

References

[1] Oscar E. Lanford, III, A computer-assisted proof of the Feigenbaum conjectures, *Bull. Amer. Math. Soc. (N.S.)* **6** (1982), 427–434.

[2] Warwick Tucker, A rigorous ODE solver and Smale's 14th problem, *Found. Comput. Math.* **2** (2002), 53–117.