Mini-Tutorial on Weak Proof Systems and Connections to SAT Solving

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Theoretical Foundations of Applied SAT Solving
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Focus of This Mini-Tutorial

Proof systems behind some current approaches to SAT solving:

- Conflict-driven clause learning — resolution
- Gröbner basis computations — polynomial calculus
- Pseudo-Boolean solvers — cutting planes

Survey (some of) what is known about these proof systems

Show some of the "benchmark formulas" used

By necessity, selective and somewhat subjective coverage — apologies in advance for omissions

Outline

1. Resolution
   - Preliminaries
   - Length, Width and Space
   - Complexity Measures and CDCL Hardness
2. Stronger Proof Systems Than Resolution
   - Polynomial Calculus
   - Cutting Planes
   - And Beyond...
3. CDCL and Efficient Proof Search

Some Notation and Terminology

- Literal a: variable x or its negation $\overline{x}$
- Clause $C = a_1 \lor \cdots \lor a_k$: disjunction of literals
  (Consider as sets, so no repetitions and order irrelevant)
- CNF formula $F = C_1 \land \cdots \land C_m$: conjunction of clauses
- k-CNF formula: CNF formula with clauses of size $\leq k$
  (where k is some constant)
- Mostly assume formulas k-CNFs (for simplicity of exposition)
  Conversion to 3-CNF (most often) doesn't change much
- N denotes size of formula (# literals, which is $\approx$ # clauses)

The Resolution Proof System

Goal: refute unsatisfiable CNF

Start with clauses of formula (axioms)

Derive new clauses by resolution rule

$$\frac{C \lor x \qquad D \lor \overline{x}}{C \lor D}$$

Refutation ends when empty clause $\perp$ derived

Can represent refutation as

- annotated list or
- DAG

Tree-like resolution if DAG is tree

1.	$x \lor y$	Axiom
2.	$x \lor \overline{y} \lor z$	Axiom
3.	$\overline{x} \lor z$	Axiom
4.	$\overline{y} \lor \overline{z}$	Axiom
5.	$\overline{x} \lor \overline{z}$	Axiom
6.	$x \lor \overline{y}$	Res$(2,4)$
7.	x	Res$(1,6)$
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9.	$\perp$	Res$(7,8)$

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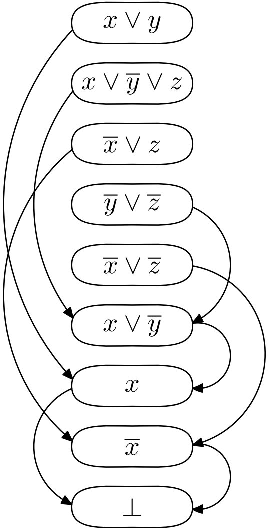

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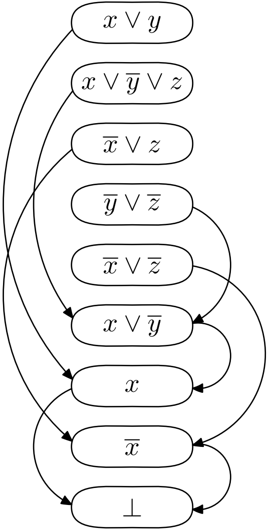

Resolution Size/Length

Size/length = # clauses in refutation

Most fundamental measure in proof complexity

Lower bound on CDCL running time

Never worse than $\exp(\mathcal{O}(N))$

Matching $\exp(\Omega(N))$ lower bounds known

Examples of Hard Formulas w.r.t Resolution Length (1/2)

Pigeonhole principle (PHP) [Hak85]

"$n+1$ pigeons don't fit into n holes"

$p_{i,1} \lor p_{i,2} \lor \cdots \lor p_{i,n}$	every pigeon i gets a hole
$\overline{p}_{i,j} \lor \overline{p}_{i',j}$	no hole j gets two pigeons

Can also add "functionality" and "onto" axioms

$\overline{p}_{i,j} \lor \overline{p}_{i,j'}$	no pigeon i gets two holes
$p_{1,j} \lor p_{2,j} \lor \cdots \lor p_{n+1,j}$	every hole j gets a pigeon

Even Onto-FPHP formula is hard for resolution

But only length lower bound $\exp\left(\Omega\left(\sqrt[3]{N}\right)\right)$ in terms of formula size

Examples of Hard Formulas w.r.t Resolution Length (2/2)

Tseitin formulas [Urq87]

"Sum of degrees of vertices in graph is even"

- Let variables = edges
- Label every vertex 0/1 so that sum of labels odd
- Write CNF requiring parity of edges around vertex = label

Requires length $\exp(\Omega(N))$ on well-connected so-called expanders

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Random k-CNF formulas [CS88]

Randomly sample Δn k-clauses over n variables
($\Delta \gtrsim 4.5$ sufficient for $k = 3$ to get unsatisfiable CNF w.h.p.)
Again lower bound $\exp(\Omega(N))$

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$$\textit{width} \leq \mathcal{O}\left(\sqrt{(\textit{formula size } N) \cdot \log(\textit{length})}\right)$$

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Theorem ([BW01])

$$\mathit{width} \leq \mathcal{O}\left(\sqrt{(\mathit{formula\ size}\ N) \cdot \log(\mathit{length})}\right)$$

Yields superpolynomial length bounds for width $\omega\left(\sqrt{N \log N}\right)$
Almost all known lower bounds on length derivable via width

Optimality of the Length-Width Lower Bound

For tree-like resolution have width $\leq \mathcal{O}(\log(\text{length}))$ [BW01]

General resolution: no length lower bounds for width $\mathcal{O}\left(\sqrt{N \log N}\right)$ — possible to tighten analysis? No!

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Ordering principles [Stå96, BG01]
"Every (partially) ordered set $\{e_1, \ldots, e_n\}$ has minimal element"

$\overline{x}_{i,j} \lor \overline{x}_{j,i}$	anti-symmetry; not both $e_i < e_j$ and $e_j < e_i$
$\overline{x}_{i,j} \lor \overline{x}_{j,k} \lor x_{i,k}$	transitivity; $e_i < e_j$ and $e_j < e_k$ implies $e_i < e_k$
$\bigvee_{1 \leq i \leq n,\, i \neq j} x_{i,j}$	e_j is not a minimal element

Can also add "total order" axioms

$x_{i,j} \lor x_{j,i}$	totality; either $e_i < e_j$ or $e_j < e_i$

Doable in length $\mathcal{O}(N)$ but needs width $\Omega\left(\sqrt[3]{N}\right)$ (3-CNF version)

Resolution Space

Space = max # clauses in memory when performing refutation

Motivated by considerations of SAT solver memory usage

Also intrinsically interesting for proof complexity

Can be measured in different ways — focus here on most common measure clause space

Space at step t: # clauses at steps $\leq t$ used at steps $\geq t$

Example: Space at step 7 ...

1.	$x \lor y$	Axiom
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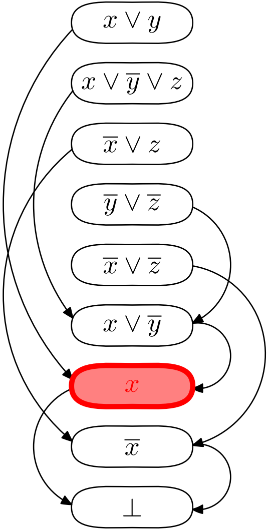

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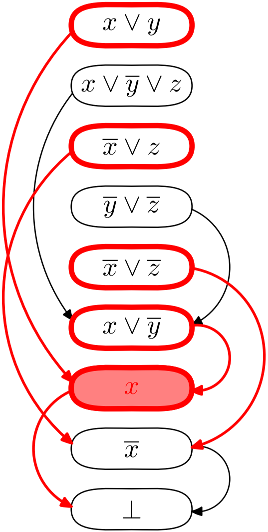
Example: Space at step 7 is 5



Bounds on Resolution Space

Space always at most $N + \mathcal{O}(1)$ [ET01]

Lower bounds for

- Pigeonhole principle [ABRW02, ET01]
- Tseitin formulas [ABRW02, ET01]
- Random k-CNFs [BG03]

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Results always matching width bounds

And proofs of very similar flavour... What is going on?

Space vs. Width

Theorem ([AD08])

$$\mathit{space} \geq \mathit{width} + \mathcal{O}(1)$$

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Pebbling formulas [BN08]

- Can be refuted in width $\mathcal{O}(1)$
- May require space $\Omega(N/\log N)$

A bit more involved to describe than previous benchmarks. . .

Pebbling Formulas: Vanilla Version

CNF formulas encoding so-called pebble games on DAGs

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
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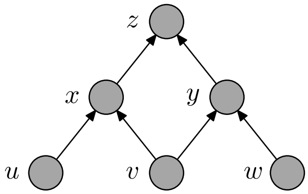

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- truth propagates upwards
- but sink is false

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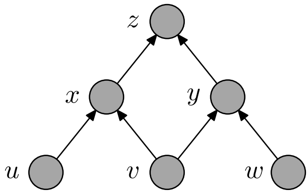

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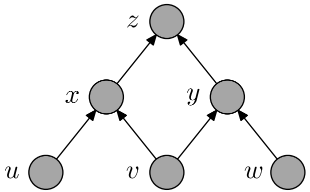

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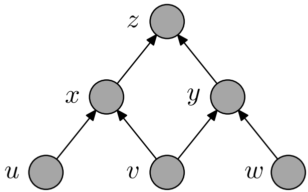

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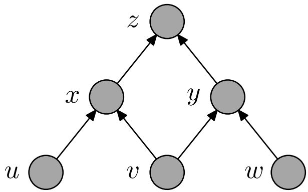

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Extensive literature on pebbling space and time-space trade-offs from 1970s and 80s

Have been useful in proof complexity before in various contexts

Hope that pebbling properties of DAG somehow carry over to resolution refutations of pebbling formulas.

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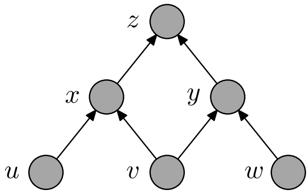

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Substituted Pebbling Formulas

Won't work — solved by unit propagation, so supereasy

Make formula harder by substituting $x_1 \oplus x_2$ for every variable x (also works for other Boolean functions with "right" properties):

$$\overline{x} \vee y$$

$$\Downarrow$$

$$\neg(x_1 \oplus x_2) \vee (y_1 \oplus y_2)$$

$$\Downarrow$$

$$\begin{aligned} & (x_1 \vee \overline{x}_2 \vee y_1 \vee y_2) \\ \wedge\ & (x_1 \vee \overline{x}_2 \vee \overline{y}_1 \vee \overline{y}_2) \\ \wedge\ & (\overline{x}_1 \vee x_2 \vee y_1 \vee y_2) \\ \wedge\ & (\overline{x}_1 \vee x_2 \vee \overline{y}_1 \vee \overline{y}_2) \end{aligned}$$

Now CNF formula inherits pebbling graph properties!

Space-Width Trade-offs

Given a formula easy w.r.t. these complexity measures, can refutations be optimized for two or more measures?

For space vs. width, the answer is a strong no

Theorem ([Ben09])

There are formulas for which

- *exist refutations in width $\mathcal{O}(1)$*
- *exist refutations in space $\mathcal{O}(1)$*
- *optimization of one measure causes (essentially) worst-case behaviour for other measure*

Holds for vanilla version of pebbling formulas

Length-Space Trade-offs

Theorem ([BN11, BBI12, BNT13])

There are formulas for which

- *exist refutations in short length*
- *exist refutations in small space*
- *optimization of one measure* causes *dramatic blow-up for other measure*

Holds for

- Substituted pebbling formulas over the right graphs
- Tseitin formulas over long, narrow rectangular grids

So no meaningful simultaneous optimization possible for length and space in the worst case

Length-Width Trade-offs?

What about length versus width?

[BW01] transforms short refutation to narrow one, but blows up length exponentially

- Is this blow-up inherent?
- Or just an artifact of the proof?

Open Problem

Are there length-width trade-offs in resolution? Or can we search for a narrow refutation and be sure to find something not significantly longer than the shortest one?

Do These Measures Say Anything About CDCL Hardness?

Recall log(length) $\lesssim$ width $\lesssim$ space

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- Searching in small width known heuristic in AI community
- Small width $\Rightarrow$ CDCL solver will provably be fast [AFT11]

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Space

- In practice, memory consumption important bottleneck
- Does space complexity correlate with hardness?

Practical Conclusions?

Experimental evaluation

- Proposed by [ABLM08]
- First(?) systematic attempt in [JMNŽ12]
- No firm conclusions — other structural properties involved?
- Ongoing work — so far both width and space seem relevant

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Broader lessons?

Performance on combinatorial benchmarks sometimes surprising

- For PHP, worse behaviour with heuristics than without
- For ordering principles, highly dependent on specific solver

Open Problem

- *Could it be interesting to explain the above phenomena?*
- *Could controlled experiments on easily scalable theoretical benchmarks yield other interesting insights?*

Polynomial Calculus (or Actually PCR)

Introduced in [CEI96]; below modified version from [ABRW02]

Clauses interpreted as polynomial equations over finite field
Any field in theory; GF(2) in practice
Example: $x \vee y \vee \overline{z}$ gets translated to $x'y'z = 0$

Derivation rules

Boolean axioms $\dfrac{}{x^2 - x = 0}$ Negation $\dfrac{}{x + x' = 1}$

Linear combination $\dfrac{p = 0 \quad q = 0}{\alpha p + \beta q = 0}$ Multiplication $\dfrac{p = 0}{xp = 0}$

Goal: Derive $1 = 0 \Leftrightarrow$ no common root $\Leftrightarrow$ formula unsatisfiable

Size, Degree and Space

Write out all polynomials as sums of monomials
W.l.o.g. all polynomials multilinear (because of Boolean axioms)

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Write out all polynomials as sums of monomials
W.l.o.g. all polynomials multilinear (because of Boolean axioms)

Size — analogue of resolution length
total # monomials in refutation (counted with repetitions)
Can also define length measure — might be much smaller

Degree — analogue of resolution width
largest degree of monomial in refutation

(Monomial) space — analogue of resolution (clause) space
max # monomials in memory during refutation (with repetitions)

Polynomial Calculus Strictly Stronger than Resolution

Polynomial calculus simulates resolution efficiently with respect to length/size, width/degree, and space simultaneously

- Can mimic resolution refutation step by step
- Hence worst-case upper bounds for resolution carry over

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- Onto functional pigeonhole principle [Rii93]

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Open Problem

Show that polynomial calculus is strictly stronger than resolution w.r.t. space

Size vs. Degree

- Degree upper bound $\Rightarrow$ size upper bound [CEI96]
  Qualitatively similar to resolution bound
  A bit more involved argument
  Again essentially tight by [ALN13]

- Degree lower bound $\Rightarrow$ size lower bound [IPS99]
  Precursor of [BW01] — can do same proof to get same bound

- Size-degree lower bound essentially optimal [GL10]
  Example: again ordering principle formulas

- Most size lower bounds for polynomial calculus derived via degree lower bounds (but machinery less developed)

Examples of Hard Formulas w.r.t. Size (and Degree)

Pigeonhole principle formulas
Follows from [AR03]
Earlier work on other encodings in [Raz98, IPS99]

Tseitin formulas with "wrong modulus"
Can define Tseitin-like formulas counting mod p for $p \neq 2$
Hard if $p \neq$ characteristic of field [BGIP01]

Random k-CNF formulas
Hard in all characteristics except 2 [BI10] (conference version '99)
Lower bound for all characteristics in [AR03]

Bounds on Polynomial Calculus Space

Lower bound for PHP with wide clauses [ABRW02]

k-CNFs much trickier — sequence of lower bounds for

- Obfuscated 4-CNF versions of PHP [FLN+12]
- Random 4-CNFs [BG13]
- Tseitin formulas on (some) expanders [FLM+13]

Open Problem

- *Prove tight space lower bounds for Tseitin on any expander*
- *Prove any space lower bound on random 3-CNFs*
- *Prove any space lower bound for any 3-CNF!?*

Space vs. Degree

Open Problem (analogue of [AD08])

Is it true that space $\geq$ degree $+ \mathcal{O}(1)$?

Partial progress: if formula requires large resolution width, then XOR-substituted version requires large space [FLM+13]

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Optimal separation of space and degree in [FLM^{+}13] by flavour of Tseitin formulas which

- can be refuted in degree $\mathcal{O}(1)$
- require space $\Omega(N)$
- but separating formulas depend on characteristic of field

Open Problem

Prove space lower bounds for substituted pebbling formulas (would give space-degree separation independent of characteristic)

Trade-offs for Polynomial Calculus

- Strong, essentially optimal space-degree trade-offs [BNT13]
  Same vanilla pebbling formulas as for resolution
  Same parameters

- Strong size-space trade-offs [BNT13]
  Same formulas as for resolution
  Some loss in parameters

Open Problem

Are there size-degree trade-offs in polynomial calculus?

Algebraic SAT Solvers?

- Quite some excitement about Gröbner basis approach to SAT solving after [CEI96]
- Promise of performance improvement failed to deliver
- Meanwhile: the CDCL revolution...
- Is it harder to build good algebraic SAT solvers, or is it just that too little work has been done (or both)?
- Some shortcut seems to be needed — full Gröbner basis computation does too much work
- Priyank Kalla will give survey talk about algebraic approaches to SAT on Tuesday

Cutting Planes

Introduced in [CCT87]

Clauses interpreted as linear inequalities over the reals with integer coefficients

Example: $x \vee y \vee \overline{z}$ gets translated to $x + y + (1 - z) \geq 1$

Derivation rules

Variable axioms $\dfrac{}{0 \leq x \leq 1}$

Multiplication $\dfrac{\sum a_i x_i \geq A}{\sum c a_i x_i \geq cA}$

Addition $\dfrac{\sum a_i x_i \geq A \quad \sum b_i x_i \geq B}{\sum (a_i + b_i) x_i \geq A + B}$

Division $\dfrac{\sum c a_i x_i \geq A}{\sum a_i x_i \geq \lceil A/c \rceil}$

Goal: Derive $0 \geq 1 \Leftrightarrow$ formula unsatisfiable

Size, Length and Space

Length = total # lines/inequalities in refutation

Size = sum also size of coefficients

Space = max # lines in memory during refutation

No (useful) analogue of width/degree

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Cutting planes

- simulates resolution efficiently w.r.t. length/size and space simultaneously
- is strictly stronger w.r.t. length/size — can refute PHP efficiently [CCT87]

Open Problem

Show cutting planes strictly stronger than resolution w.r.t. space

Hard Formulas w.r.t Cutting Planes Length

Clique-coclique formulas [Pud97]
"A graph with a k-clique is not $(k-1)$-colourable"
Lower bound via interpolation and circuit complexity

Open Problem

Prove cutting planes length lower bounds

- *for Tseitin formulas*
- *for random k-CNFs*
- *for any formula using other technique than interpolation*

Hard Formulas w.r.t Cutting Planes Space?

No space lower bounds known except conditional ones

All short cutting planes refutations of

- Tseitin formulas on expanders require large space [GP13]
  (But such short refutations probably don't exist anyway)

- (some) pebbling formulas require large space [HN12, GP13]
  (and such short refutations do exist; hard to see how exponential length could help bring down space)

Above results obtained via communication complexity

No (true) length-space trade-off results known
Although results above can also be phrased as trade-offs

Geometric SAT Solvers?

- Some work on pseudo-Boolean solvers using (subset of) cutting planes
- Seems hard to make competitive with CDCL on CNFs
- One key problem to recover cardinality constraints
- Daniel Le Berre will give survey talk about geometric approaches to SAT on Tuesday

Semialgebraic Proof Systems

- Proof systems using polynomial inequalities over the reals
- Kind of a combination/generalization of polynomial calculus and cutting planes
- Used to reason about (near-)optimality of combinatorial optimization
- Albert Atserias will give a separate mini-tutorial about semialgebraic proof systems on Tuesday

How Efficient Resolution Refutations Can CDCL Find?

DPLL (no clause learning)

Always yields tree-like refutations

Exponentially weaker than general resolution in worst case

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CDCL

Generates DAG-like refutations, but with very particular structure

- Clauses derived by "input resolution" w.r.t. clause database
- Learned clauses should be asserting
- Derivations look locally regular w.r.t. clause database
  (only resolve on each variable once along path)

Can CDCL be as efficient as general, unrestricted resolution?

How Measure Efficiency? CDCL as a Proof System

1. **Automatizability**
   - Run in time polynomial in smallest possible refutation
   - Seems too strict a requirement even for resolution [AR08]

2. **More relaxed notion**
   - Can CDCL run in time polynomial in smallest possible refutation assuming that all free decisions are made optimally?
   - I.e., does CDCL polynomially simulate resolution viewed as a proof system?
   - Intuitively: No worst-case guarantees, but promise to work well if one can get heuristics right

CDCL Polynomially Simulates Resolution

Answer: yes, polynomial simulation! [BKS04, BHJ08, HBPV08]
But with varying restrictions on model:

- Non-standard learning schemes
- Decisions flipping propagated variables
- Decisions past conflicts
- Preprocessing of formula (with new variables)

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Natural model of CDCL polynomially simulates resolution

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Theorem ([PD11])

Natural model of CDCL polynomially simulates resolution

Theorem ([AFT11])

If in addition resolution width is small, then CDCL solver with enough randomness will find good refutation with high probability

Assumptions Behind Effectiveness of CDCL

1. **Frequent restarts**
   How efficient is CDCL without restarts?
   Can it simulate resolution or not?

2. **Never forget clauses**
   Not how CDCL solvers actually operate
   Just technical condition or necessary for proofs to go through?

3. **Randomness**
   Not used much in practice
   Seems necessary for theoretical results in [AFT11]

Further Questions About CDCL Proof System

- Possible to get more "syntactic" description of proof system in [AFT11, PD11]? (Now more like execution trace of solver)
- Can one model (clause database) space in such a proof system in some nice way?
- Do upper and lower bounds and trade-offs results carry over from general resolution?

Summing up

- Survey of resolution, polynomial calculus and cutting planes
- Resolution fairly well understood
- Polynomial calculus less so
- Cutting planes almost not at all
- Could there be interesting connections between proof complexity measures and hardness of SAT?
- How can we build efficient SAT solvers on stronger proof systems than resolution?

Summing up

- Survey of resolution, polynomial calculus and cutting planes
- Resolution fairly well understood
- Polynomial calculus less so
- Cutting planes almost not at all
- Could there be interesting connections between proof complexity measures and hardness of SAT?
- How can we build efficient SAT solvers on stronger proof systems than resolution?

Thank you for your attention!

References I

- Carlos Ansótegui, María Luisa Bonet, Jordi Levy, and Felip Manyà. Measuring the hardness of SAT instances. In *Proceedings of the 23rd National Conference on Artificial Intelligence (AAAI '08)*, pages 222–228, 2008.

- Michael Alekhnovich, Eli Ben-Sasson, Alexander A. Razborov, and Avi Wigderson. Space complexity in propositional calculus. *SIAM Journal on Computing*, 31(4):1184–1211, 2002. Preliminary version in *STOC '00*.

- Albert Atserias and Víctor Dalmau. A combinatorial characterization of resolution width. *Journal of Computer and System Sciences*, 74(3):323–334, 2008. Preliminary version in *CCC '03*.

- Albert Atserias, Johannes Klaus Fichte, and Marc Thurley. Clause-learning algorithms with many restarts and bounded-width resolution. *Journal of Artificial Intelligence Research*, 40:353–373, 2011. Preliminary version in *SAT '09*.

- Albert Atserias, Massimo Lauria, and Jakob Nordström. Narrow proofs may be maximally long. Submitted, 2013.

References II

- Michael Alekhnovich and Alexander A. Razborov. Lower bounds for polynomial calculus: Non-binomial case. *Proceedings of the Steklov Institute of Mathematics*, 242:18–35, 2003. Preliminary version in *FOCS '01*.
- Michael Alekhnovich and Alexander A. Razborov. Resolution is not automatizable unless W[P] is tractable. *SIAM Journal on Computing*, 38(4):1347–1363, 2008. Preliminary version in *FOCS '01*.
- Paul Beame, Chris Beck, and Russell Impagliazzo. Time-space tradeoffs in resolution: Superpolynomial lower bounds for superlinear space. In *Proceedings of the 44th Annual ACM Symposium on Theory of Computing (STOC '12)*, pages 213–232, 2012.
- Eli Ben-Sasson. Size-space tradeoffs for resolution. *SIAM Journal on Computing*, 38(6):2511–2525, 2009. Preliminary version in *STOC '02*.
- María Luisa Bonet and Nicola Galesi. Optimality of size-width tradeoffs for resolution. *Computational Complexity*, 10(4):261–276, 2001. Preliminary version in *FOCS '99*.

References III

Eli Ben-Sasson and Nicola Galesi. Space complexity of random formulae in resolution. *Random Structures and Algorithms*, 23(1):92–109, 2003. Preliminary version in *CCC '01*.

Ilario Bonacina and Nicola Galesi. Pseudo-partitions, transversality and locality: A combinatorial characterization for the space measure in algebraic proof systems. In *Proceedings of the 4th Conference on Innovations in Theoretical Computer Science (ITCS '13)*, pages 455–472, 2013.

Samuel R. Buss, Dima Grigoriev, Russell Impagliazzo, and Toniann Pitassi. Linear gaps between degrees for the polynomial calculus modulo distinct primes. *Journal of Computer and System Sciences*, 62(2):267–289, 2001. Preliminary version in *CCC '99*.

Samuel R. Buss, Jan Hoffmann, and Jan Johannsen. Resolution trees with lemmas: Resolution refinements that characterize DLL-algorithms with clause learning. *Logical Methods in Computer Science*, 4(4:13), 2008.

Eli Ben-Sasson and Russell Impagliazzo. Random CNF's are hard for the polynomial calculus. *Computational Complexity*, 19:501–519, 2010. Preliminary version in *FOCS '99*.

References IV

Paul Beame, Henry Kautz, and Ashish Sabharwal. Towards understanding and harnessing the potential of clause learning. *Journal of Artificial Intelligence Research*, 22:319–351, 2004.

Eli Ben-Sasson and Jakob Nordström. Short proofs may be spacious: An optimal separation of space and length in resolution. In *Proceedings of the 49th Annual IEEE Symposium on Foundations of Computer Science (FOCS '08)*, pages 709–718, 2008.

Eli Ben-Sasson and Jakob Nordström. Understanding space in proof complexity: Separations and trade-offs via substitutions. In *Proceedings of the 2nd Symposium on Innovations in Computer Science (ICS '11)*, pages 401–416, 2011.

Chris Beck, Jakob Nordström, and Bangsheng Tang. Some trade-off results for polynomial calculus. In *Proceedings of the 45th Annual ACM Symposium on Theory of Computing (STOC '13)*, pages 813–822, 2013.

Eli Ben-Sasson and Avi Wigderson. Short proofs are narrow—resolution made simple. *Journal of the ACM*, 48(2):149–169, 2001. Preliminary version in *STOC '99*.

References V

- William Cook, Collette Rene Coullard, and Gyorgy Turán. On the complexity of cutting-plane proofs. *Discrete Applied Mathematics*, 18(1):25–38, 1987.
- Matthew Clegg, Jeffery Edmonds, and Russell Impagliazzo. Using the Groebner basis algorithm to find proofs of unsatisfiability. In *Proceedings of the 28th Annual ACM Symposium on Theory of Computing (STOC '96)*, pages 174–183, 1996.
- Vašek Chvátal and Endre Szemerédi. Many hard examples for resolution. *Journal of the ACM*, 35(4):759–768, 1988.
- Juan Luis Esteban and Jacobo Torán. Space bounds for resolution. *Information and Computation*, 171(1):84–97, 2001. Preliminary versions of these results in *STACS '99* and *CSL '99*.
- Yuval Filmus, Massimo Lauria, Mladen Mikša, Jakob Nordström, and Marc Vinyals. Towards an understanding of polynomial calculus: New separations and lower bounds. In *Proceedings of the 40th International Colloquium on Automata, Languages and Programming (ICALP '13)*, pages 437–448. 2013.

References VI

Yuval Filmus, Massimo Lauria, Jakob Nordström, Neil Thapen, and Noga Ron-Zewi. Space complexity in polynomial calculus. In *Proceedings of the 27th Annual IEEE Conference on Computational Complexity (CCC '12)*, pages 334–344, 2012.

Nicola Galesi and Massimo Lauria. Optimality of size-degree trade-offs for polynomial calculus. *ACM Transactions on Computational Logic*, 12:4:1–4:22, 2010.

Mika Göös and Toniann Pitassi. Communication lower bounds via critical block sensitivity. Technical Report 1311.2355, arXiv.org, 2013.

Armin Haken. The intractability of resolution. *Theoretical Computer Science*, 39(2-3):297–308, 1985.

Philipp Hertel, Fahiem Bacchus, Toniann Pitassi, and Allen Van Gelder. Clause learning can effectively P-simulate general propositional resolution. In *Proceedings of the 23rd National Conference on Artificial Intelligence (AAAI '08)*, pages 283–290, 2008.

References VII

Trinh Huynh and Jakob Nordström. On the virtue of succinct proofs: Amplifying communication complexity hardness to time-space trade-offs in proof complexity. In *Proceedings of the 44th Annual ACM Symposium on Theory of Computing (STOC '12)*, pages 233–248, 2012.

Russell Impagliazzo, Pavel Pudlák, and Jiri Sgall. Lower bounds for the polynomial calculus and the Gröbner basis algorithm. *Computational Complexity*, 8(2):127–144, 1999.

Matti Järvisalo, Arie Matsliah, Jakob Nordström, and Stanislav Živný. Relating proof complexity measures and practical hardness of SAT. In *Proceedings of the 18th International Conference on Principles and Practice of Constraint Programming (CP '12)*, pages 316–331. 2012.

Knot Pipatsrisawat and Adnan Darwiche. On the power of clause-learning SAT solvers as resolution engines. *Artificial Intelligence*, 175:512–525, 2011. Preliminary version in *CP '09*.

Pavel Pudlák. Lower bounds for resolution and cutting plane proofs and monotone computations. *Journal of Symbolic Logic*, 62(3):981–998, 1997.

References VIII

- Alexander A. Razborov. Lower bounds for the polynomial calculus. *Computational Complexity*, 7(4):291–324, 1998.
- Søren Riis. *Independence in Bounded Arithmetic*. PhD thesis, University of Oxford, 1993.
- Gunnar Stålmarck. Short resolution proofs for a sequence of tricky formulas. *Acta Informatica*, 33(3):277–280, 1996.
- Alasdair Urquhart. Hard examples for resolution. *Journal of the ACM*, 34(1):209–219, 1987.