Lower bounds for width-restricted clause learning

Jan Johannsen

Institut für Informatik
LMU München

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partially based on joint work with
Sam Buss, Jan Hoffmann & Eli Ben-Sasson
Resolution Trees with Lemmas

A Resolution tree with lemmas (RTL) for formula $F$ is an ordered binary tree labelled with clauses s.t.
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- $C_{\text{root}} = \Box$

- if $v$ has 2 children $u$ and $u'$, then $C_v = \text{Res}(C_u, C_{u'})$ for some variable $x$

- if $v$ has 1 child $u$, then $C_v \supseteq C_u$

- if $v$ is a leaf, then $C_v \in F$ or $C_v = C_u$ for some $u \prec v$ (lemma)

$\prec$ is the post-order on trees.
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  $$\mathcal{C}_v = \text{Res}_x(\mathcal{C}_u, \mathcal{C}_{u'})$$
  for some variable $x$

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- if $v$ is a leaf, then
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Clause learning and \textit{RTL}

**Theorem (Buss, Hoffmann, JJ)**

If unsatisfiable formula $F$ is refuted by $\text{DPLL+CL}$ in $s$ steps, then $F$ has an RTL-refutation $R$ of size $s \cdot n^{O(1)}$. 
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*Moreover, the lemmas used in $R$ are among the clauses learned by the algorithm.*
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\textit{Moreover, the lemmas used in $R$ are among the clauses learned by the algorithm.}

In fact, the paper defines a subsystem $WRTI < RTL$ for which also the converse holds.
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Theorem (Buss, Hoffmann, JJ)

*If unsatisfiable formula $F$ is refuted by $DPLL+CL$ in $s$ steps, then $F$ has an $RTL$-refutation $R$ of size $s \cdot n^{O(1)}$. Moreover, the lemmas used in $R$ are among the clauses learned by the algorithm.*

In fact, the paper defines a subsystem $WRTI < RTL$ for which also the converse holds.

*Here:* lower bounds for $RTL(k)$:

A refutation $R$ in $RTL$ is in $RTL(k)$, if every lemma $C$ used in $R$ is of width $w(C) \leq k$. 


Complexity of the Pigeonhole Principle

Theorem (Haken 1985)

Resolution proofs of PHP$_n$ require size $2^{\Omega(n)}$. 

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Tree-like resolution proofs of PHPₙ require size $2^{\Omega(n \log n)}$. 

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\[ \text{Every RTL}(n/2)\text{-refutation of } PHP_n \text{ is of size } 2^{\Omega(n \log n)}. \]

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Every RTL($n/2$)-refutation of $PHP_n$ is of size $2^\Omega(n \log n)$.

- Let $R$ be a refutation of $PHP_n$
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- Pick $\rho$ with $C[\rho] = 0$
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Every RTL\((n/2)\)-refutation of \(\text{PHP}_n\) is of size \(2^{\Omega(n \log n)}\).

- Let \(R\) be a refutation of \(\text{PHP}_n\)
- Find first \(C\) with \(w(C) \leq k\)
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- $R_C \models \rho$ is refutation of $PHP_n \models \rho$
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  \[ PHP_n \models \rho = PHP_{n-|\rho|} \]
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Every $RTL(n/2)$-refutation of $PHP_n$ is of size $2^{\Omega(n \log n)}$.

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**Main Lemma:** For $C$ in $R$ with $w(C) \leq k$, there is a matching restriction $\rho$ with $C \rho = 0$ and $|\rho| \leq k$
The Ordering Principle

...says: An ordering of \([n]\) has a maximum
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The formula \(\text{Ord}_n\):

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The formula \(Ord_n\):

- **variables** \(x_{i,j}\) for \(i, j \leq n\) and \(i \neq j\)

- **totality clauses** \(x_{i,j} \lor x_{j,i}\) for all \(i, j\)

- **asymmetry clauses** \(\overline{x}_{i,j} \lor \overline{x}_{j,i}\)

- **transitivity clauses** \(\overline{x}_{i,j} \lor \overline{x}_{j,k} \lor \overline{x}_{k,i}\)

- **maximum clauses** \(\lor j \neq i x_{i,j}\) for all \(i\)
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- **maximum clauses** \(\lor_{j \neq i} x_{i,j}\) for all \(i\)
Complexity of the Ordering Principle

Theorem (Stålmarck 1997)

There are regular resolution proofs of $\text{Ord}_n$ of size $O(n^3)$. 
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Theorem (Bonet, Galesi 1999)

Tree-like resolution proofs of $\text{Ord}_n$ require size $2^{\Omega(n)}$. 
Cyclic clauses

For clause $C$, the graph $G(C)$ has edges

$$(i, j) \quad \text{for } \overline{x}_{i,j} \in C \quad \text{and} \quad (j, i) \quad \text{for } x_{i,j} \in C$$
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Definition: $C$ is cyclic, if $G(C)$ contains a cycle.
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Definition: $C$ is cyclic, if $G(C)$ contains a cycle.

Lemma: A cyclic clause $C$ has a tree-like resolution derivation from $Ord_n$ of size $O(w(C))$. 
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**Definition:** $C$ is **cyclic**, if $G(C)$ contains a cycle.

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The main lemmas

Lemma

If there is an $RTL(k)$-refutation of $Ord_n$ of size $s$, then there is another one using no cyclic lemmas of size $O(sk)$. 

Proof: Replace each cyclic lemma by its derivation of size $O(k)$. 

Lemma

If $C$ is acyclic with $w(C) \leq k$, then there is an ordering restriction $\sigma$ with $|\sigma| \leq 2^k$ such that $C |\sigma| = 0$. 

Proof: For $C$ acyclic $G(C)$ is a dag; obtain $\sigma$ as a topological ordering of $G(C)$. 
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*If there is an RTL*\(_k\)*-refutation of* \(\text{Ord}_n\) *of size* \(s\), *then there is another one using no cyclic lemmas of size* \(O(sk)\).

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*If* \(C\) *is acyclic with* \(w(C) \leq k\), *then there is an ordering restriction* \(\sigma\) *with* \(|\sigma| \leq 2k\) *such that* \(C|\sigma = 0\).
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Lemma

If there is an RTL($k$)-refutation of $\text{Ord}_n$ of size $s$, then there is another one using no cyclic lemmas of size $O(sk)$.

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Lemma

If $C$ is acyclic with $w(C) \leq k$, then there is an ordering restriction $\sigma$ with $|\sigma| \leq 2k$ such that $C \models \sigma = 0$.

Proof: For $C$ acyclic $G(C)$ is a dag

$\leadsto$ obtain $\sigma$ as a topological ordering of $G(C)$.
The lower bound

**Theorem**

For $k < n/4$, every $RTL(k)$-refutation of $Ord_n$ is of size $2^{\Omega(n)}$. 
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Let $R$ be a refutation of $\text{Ord}_n$.
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- $R_C \models \sigma$ is refutation of $\text{Ord}_n \models \sigma$
The lower bound

**Theorem**

*For* \( k < n/4 \), every RTL\((k)\)-refutation of Ord\(_n\) is of size \( 2^{\Omega(n)} \).

- Let \( R \) be a refutation of Ord\(_n\)
- Remove cyclic lemmas
- Find first \( C \) with \( w(C) \leq k \)
- Subtree \( R_C \) is tree-like derivation of \( C \)
- Pick \( \sigma \) with \( C \models \sigma = 0 \)
- \( R_C \models \sigma \) is refutation of Ord\(_n\)\(\models \sigma \)
- \( \text{Ord}_n \models \sigma = \text{Ord}_n - |\sigma| + 1 \)
**The lower bound**

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- lower bound by Bonet/Galesi
A Game

Let $X$ be a set of variables, and $w \leq |X|$. 
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A \textit{w-system of restrictions} over $X$ is $\mathcal{H} \neq \emptyset$ with
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Let $X$ be a set of variables, and $w \leq |X|$. A $w$-system of restrictions over $X$ is $\mathcal{H} \neq \emptyset$ with

- $|\rho| \leq w$ for $\rho \in \mathcal{H}$,
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Let $X$ be a set of variables, and $w \leq |X|$.

A $w$-system of restrictions over $X$ is $\mathcal{H} \neq \emptyset$ with

- $|\rho| \leq w$ for $\rho \in \mathcal{H}$,
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- extension property: if $\rho \in \mathcal{H}$ with $|\rho| < w$, and $v \in X \setminus \text{dom} \rho$, then there is $\rho' \supseteq \rho$ in $\mathcal{H}$ that sets $v$. 
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$\mathcal{H}$ avoids $C$ if $C[\rho] \neq 0$ for all $\rho \in \mathcal{H}$.
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$\mathcal{H}$ avoids $C$ if $C| \rho \neq 0$ for all $\rho \in \mathcal{H}$

$\mathcal{H}$ avoids $F$ if $\mathcal{H}$ avoids all $C \in F$
Resolution width and systems of restrictions

Theorem (Atserias & Dalmau)

\( F \) requires resolution width \( w \) iff there is a \( w \)-system of restrictions that avoids \( F \).
Resolution width and systems of restrictions

Theorem (Atserias & Dalmau)

F requires resolution width w iff there is a w-system of restrictions that avoids F.

Theorem (Ben-Sasson & Wigderson)

If a d-CNF formula F requires resolution width w, then tree-like resolution proofs of F require size $2^{w-d}$. 
Restricted systems

Lemma

Let $\mathcal{H}$ be a $w$-system of restrictions over $X$, and $\rho \in \mathcal{H}$.

$$\mathcal{H}[\rho] := \{ \sigma ; \text{dom} \sigma \subseteq X \setminus \text{dom} \rho \text{ and } \sigma \cup \rho \in \mathcal{H} \text{ and } |\sigma| \leq w - |\rho| \}$$

is a $w - |\rho|$ system of restrictions over $X \setminus \text{dom} \rho$
Restricted systems

Lemma
Let $\mathcal{H}$ be a $w$-system of restrictions over $X$, and $\rho \in \mathcal{H}$.

$$\mathcal{H}\mid \rho := \{ \sigma ; \text{dom} \sigma \subseteq X \setminus \text{dom} \rho \text{ and } \sigma \cup \rho \in \mathcal{H} \text{ and } |\sigma| \leq w - |\rho| \}$$

is a $w - |\rho|$ system of restrictions over $X \setminus \text{dom} \rho$

Lemma
If $\mathcal{H}$ avoids $F$, then $\mathcal{H}\mid \rho$ avoids $F\mid \rho$. 
The general lower bound

Theorem

If $F$ requires resolution width $w$, then every $RTL(k)$-refutation of $F$ is of size $2^w - 2^k$. 
The general lower bound

Theorem

If $F$ requires resolution width $w$, then every $RTL(k)$-refutation of $F$ is of size $2^{w-2k}$.

- Let $R$ be a refutation of $F$. 
The general lower bound

Theorem

*If F requires resolution width w, then every RTL(k)-refutation of F is of size $2^{w-2k}$.*

- Let $R$ be a refutation of $F$.
- Find first $C$ with $w(C) \leq k$ not avoided by $\mathcal{H}$
The general lower bound

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If $F$ requires resolution width $w$, then every $RTL(k)$-refutation of $F$ is of size $2^{w-2k}$.

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References:
- Ben-Sasson & Wigderson
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- $R_C[\rho]$ is of size $2^{w-2k}$ by Ben-Sasson & Wigderson
Application

\[ E_3(F) := \text{3-CNF expansion of } F \]

Theorem (Bonet, Galesi)

\[ E_3(\text{Ord}_n) \text{ requires resolution width } n/6. \]
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**Theorem (Bonet, Galesi, JJ)**

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**Theorem (Bonet, Galesi, JJ)**

\[ E_3(\text{Ord}_n) \text{ requires resolution width } n/2. \]

**Corollary**

Every RTL\((n/6)\)-refutation of \(E_3(\text{Ord}_n)\) is of size \(2^{n/6}\).
Application

$E_3(F) := 3$-CNF expansion of $F$

Theorem (Bonet, Galesi, JJ)
$E_3(\text{Ord}_n)$ requires resolution width $n/2$.

Corollary
Every RTL$(n/6)$-refutation of $E_3(\text{Ord}_n)$ is of size $2^{n/6}$.

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Every RTL$(n/6)$-refutation of $\text{Ord}_n$ is of size $2^{n/6-\log n}$.
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For every $k$, there is a family of formulas $F_n^{(k)}$ such that
A Hierarchy

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For every \( k \), there is a family of formulas \( F_n^{(k)} \) such that

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Even regular, without weakening.
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For every $k$, there is a family of formulas $F_n^{(k)}$ such that

- $F_n^{(k)}$ have $RTL(k + 1)$-refutations of size $n^{O(1)}$. Even regular, without weakening.

- $F_n^{(k)}$ requires $RTL(k)$-refutations of size $2^{\Omega(n/\log n)}$. This even holds for $k = k(n)$ when $k(n) = O(\log n)$. 