Hodge Theory for HI cohomology

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Overview

The classical Hodge Theorem

Theorem

Let M be a compact Riemannian manifold with metric g. Then

$$\mathcal{H}^k(M,g) \cong H^k_{\mathrm{dR}}(M),$$

where

- $\mathcal{H}^k(M,g)$ *denotes the kernel of the Laplacian on* L^2 *differential forms over* M,
- $H^k_{\mathrm{dR}}(M)$ *denotes the deRham cohomology of* M, *which we may calculate from the complex* $(\Omega^*(M), d)$.

General Hodge theorems

Let X be a smooth manifold that is the interior of a manifold with corners (i.e., with finite topology). A generalisation of the Hodge theorem is an isomorphism of the form:

Kernel of the Laplacian for some metric on some set of forms over X	$\cong$	Cohomology of a complex defined usually over a related space, $\widehat{X}$.

This talk will be about a Hodge theorem for a new cohomology theory, called HI theory, which is related to intersection cohomology (IH), and has been defined by M. Banagl, my collaborator on this work.

Pseudomanifolds

A pseudomanifold X of dimension n

- is a stratified space, $X_0 \subset X_1 \subset \cdots X_{n-2} = X_{n-1} \subset X_n$,
- where $X_k - X_{k-1}$ is a smooth open manifold of dimension k,
- where around any point on $X_k - X_{k-1}$ there is a neighborhood of the form $B_k \times C(L_k)$ where L_k is a compact pseudo manifold of dimension $n-k-1$, (the *link* at the point)
- and where X has a locally finite covering by neighbourhoods of this type.

The regular stratum $X - X_{n-1}$ is the interior of a manifold with corners, $\hat{X}$. If there are only two strata, $\hat{X}$ is just the complement of a tubular neighborhood of the singular stratum, $X_{n-1} := \Sigma$.

Intersection Cohomology: a definition

Intersection cohomologies are parametrised by a vector $\overline{p}$ consisting of integers $\overline{p}(n-k)$. The intersection cohomology $IH^*_{\overline{p}}(X)$ is the hypercohomology of any complex of fine sheaves on X that satisfies the following Poincaré lemma on the local neighbourhoods:

$$IH^j_{\overline{p}}(B_k \times C(L_k)) = \begin{cases} IH^j_{\overline{p}}(L_k) & j < c = n-k-1-\overline{p}(n-k) \\ 0 & j > c. \end{cases}$$

E.g., if there are only two strata, then we can use the complex

$$\Omega^*_{\overline{p}}(X) := \{\omega \in \Omega^*(\hat{X}) \mid inc^*\omega(V_1, \cdots, V_c) = 0, \, \pi_*(V_i) = 0\},$$

where

- $inc : \partial \hat{X} \to \hat{X}$,
- $\pi : \partial \hat{X} \to \Sigma$ is the *link bundle*.

Poincaré duality and Witt spaces

Dual perversities: each perversity, $\overline{p}$ (that satisfies a certain set of conditions), has a *dual perversity* $\overline{q}$ so that $IH^{j}_{\overline{p}}(X) \cong IH^{n-j}_{\overline{q}}(X)$.

Middle perversity when there exists an admissible perversity, $\overline{m}$ that is its own dual, we call $\overline{m}$ the middle perversity. Then $IH^{*}_{\overline{m}}(X)$ satisfies Poincaré duality with itself.

Witt spaces This occurs in particular if $IH^{k/2}_{\overline{m}}(L_k) = 0$, where L_k is a link of dimension k in X. This is a condition that recursively ensures the middle perversity IH is defined for each link, so this makes sense. Such an X is called a Witt space.

Conical metrics on Witt spaces

A conical metric on a neighbourhood of the form $B_k \times C(L)$ is an incomplete metric of the form

$$g_{cone} = dx^2 + x^2 ds^2_{L_k} + ds^2_{B_k}, \qquad x \geq 0$$

where $ds^2_{L_k}$ is a conical metric on the pseudo manifold L_k. Thus recursively and by this local description we may define a conical metric on any pseudomanifold, X.

Theorem

(Cheeger, 1980's; ALMP 2013) If X is a compact Witt space, then the Laplacian on differential forms over X with respect to a conical metric has a unique self-adjoint extension to L^2 forms, and:

$$\mathcal{H}^j(\hat{X}, g_{cone}) \cong IH^j_{\overline{m}}(X).$$

Cusp metrics on Witt spaces

An iterated cusp metric on a neighbourhood of the form $B_k \times C(L_k)$ is a complete metric of the form

$$g_{cusp} = \frac{dx^2}{x^2} + x^2 ds^2_{L_k} + ds^2_{B_k}, \qquad x \geq 0$$

where ds^2_L is an iterated cusp metric on the pseudomanifold L_k. Thus recursively and by this local description we may define a conical metric on any pseudomanifold, X.

Theorem

(H- and F. Rochon, 2011) If X is a compact Witt space, then the Laplacian on differential forms over X with respect to an iterated cusp metric is Fredholm as a map between Sobolev spaces of forms, and:

$$\mathcal{H}^j(\hat{X}, g_{cusp}) \cong IH^j_{\overline{m}}(X).$$

Fibred cusp and fibred boundary metrics

Less is known in general if X is not Witt. In this case, there is not a middle perversity. However there are perversities close to the middle that are dual to one another called the upper ($\overline{m}$) and lower ($\underline{m}$) middle perversities.

If X has only two strata, $X_k \subset X_n$, then we get the following:

Theorem

(Hausel, H- and Mazzeo, 2005) Let X be a compact pseudomanifold with two strata. Then

$$\mathcal{H}^*(X_n - X_k, g_{cusp}) \cong \mathrm{Im}(\mathrm{IH}^*_{\underline{m}}(X) \to \mathrm{IH}^*_{\overline{m}}(X))$$

Also: Banagl, Mazzeo, Piazza, Albin and Leichtnam: Hodge theory for pseudomanifolds with Lagrangian structures on link cohomology bundles.

A new cohomology theory: $HI(X)$

Recall: intersection cohomology has *upper truncation on cohomology*, i.e, Poincaré lemma:

$$IH^j_{\bar{p}}(B_k \times C(L)) = IH^j_{\bar{p}}(L) \text{ if } j < c, \text{ and 0 otherwise.}$$

The idea of HI theory (Banagl): cohomology theory for pseudomanifolds defined through a *lower truncation* on cohomology, i.e., Poincaré lemma of the form:

$$HI^j_{\bar{p}}(B_k \times C(L)) = HI^j_{\bar{p}}(L) \text{ if } j \geq c, \text{ and 0 otherwise.}$$

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Motivations from mathematics: such a theory would give a graded ring structure on the cohomology for a single perversity, not just for all perversities at the same time.
Motivations from string theory: related to mirror symmetry and preservation of signature across conifold transitions.

Original construction of $HI^*_{\overline{p}}(X)$

HI theory was first defined by adding cells to $\hat{X}$ to construct, for each perversity, $\overline{p}$, a new cw-complex $I^{\overline{p}}X$ called the *intersection space* of perversity $\overline{p}$ associated to X.

Then by definition $HI_*^{\overline{p}}(X) := H_*(I^{\overline{p}}X)$.

Basic idea: add cells to kill lower homology classes in link.

Carried out so far for:

- pseudomanifolds with isolated singularities, and
- pseudomanifolds with two strata where the link bundle of the singular stratum carries a local product metric.

HI deRham complex

Let X be a pseudomanifold with two strata such that the unit normal bundle of the singular stratum carries a local product metric. Let L denote the (smooth) link of this bundle. Define

$$\tau_{\overline{p}}\Omega^j(L) := \begin{cases} \Omega^j(L) & j > \overline{p}(k) \\ \mathrm{Ker}(\delta) & j = \overline{p}(k) \\ 0 & j < \overline{p}(k). \end{cases}$$

Use the local product metric on $\partial\hat{X}$ to extend this to $\tau_{\overline{p}}\Omega^j(\partial\hat{X})$

Now define a subcomplex of smooth forms over $\hat{X}$:

$$CI^*_{\overline{p}}(X) := \{\omega \in \Omega^*(\partial\hat{X}) \mid inc^*\omega \in \tau_{\overline{p}}\Omega^j(\partial\hat{X}),$$

$$inc^*(d\omega) \in \tau_{\overline{p}}\Omega^{j+1}(\partial\hat{X})\}.$$

HI deRham theorem

Theorem

(Banagl; Essig; Banagl, H-) Let X be a pseudomanifold with two strata such that the unit normal bundle of the singular stratum is a product $\Sigma \times L$. Then the cohomology of the complex $(CI^*_{\overline{p}}(X), d)$ is dual to $HI^{\overline{p}}_*(X) := H_*(I^{\overline{p}}X)$.

Global *HI* cohomology for isolated singularities

If X is a pseudomanifold with isolated singularities, then $HI^j_{\overline{p}}(X)$ can be described in terms of standard cohomologies as follows:

$$HI^j_{\overline{p}}(X) \cong \begin{cases} H^j(X, \Sigma) & j < \overline{p}(0) \\ G^j(X, \Sigma) & j = \overline{p}(0) \\ H^j(X - \Sigma) & j > \overline{p}(0). \end{cases}$$

where

$$G^j(X, \Sigma) \cong \frac{H^j(X, \Sigma) \oplus H^j(X)}{\mathrm{Im}(H^j(X, \Sigma) \to H^j(X))}.$$

Compare this to

$$IH^j_{\overline{p}}(X) \cong \begin{cases} H^j(X - \Sigma) & j < \overline{p}(0) \\ \mathrm{Im}(H^j(X, \Sigma) \to H^j(X)) & j = \overline{p}(0) \\ H^j(X, \Sigma) & j > \overline{p}(0). \end{cases}$$

Extended b-harmonic forms

The space

$$G^j(X,\Sigma) \cong \frac{H^j(X,\Sigma) \oplus H^j(X)}{\mathrm{Im}(H^j(X,\Sigma) \to H^j(X))}.$$

is known to arise in Hodge theory for manifolds with b-metrics (infinite cylindrical ends). In fact, there is the following theorem

Theorem

(Melrose) Let (M, g_b) be a manifold with a b-metric where M is the interior of a manifold with boundary $\overline{M}$. Let

$$\mathcal{H}^*_{ext}(M, g_b) := \bigcap_{\epsilon>0} \{\omega \in x^{-\epsilon} L^2_b \Omega^*_b(M) \mid (d+\delta)\omega = 0\}$$

Then

$$\mathcal{H}^*_{ext}(M, g_b) \cong G^*(\overline{M}, \partial\overline{M})).$$

Extended weighted harmonic forms

To state the *HI* Hodge theorem for X with product link bundle, we need to adapt the definition of $\mathcal{H}^*_{ext}(M, g_b)$.

- use a *conical fibre* metric instead of a b-metric. Near Σ,

$$g_{cf} := \frac{dx^2}{x^4} + \pi^* ds^2_\Sigma + \frac{ds^2_F}{x^2}.$$

- add a weight: $x^c L^2_{cf}$ instead of L^2_b
- use forms in the kernel of $d + \delta_c$, where δ_c is the adjoint of d with respect to the pairing on $x^c L^2_{cf} \Omega^*_{cf}$.

Define

$$\mathcal{H}^*_{ext}(M, g_{cf}, c) = \bigcap_{\epsilon > 0} \{\omega \in x^{-\epsilon + c} L^2_{cf} \Omega^*_{cf}(M) \mid (d + \delta_c)\omega = 0\}.$$

HI Hodge theorem for isolated singularities

Theorem

(Banagl, H-) Let X be a compact n-dimensional pseudomanifold with two strata $\Sigma \subset X$, and a product link bundle, where $\dim(L) = l$. Let g_{cf} be a conical fibre metric on $X - \Sigma$. Then

$$HI^*_p(X) \cong \mathcal{H}^*_{ext}\left(X - \Sigma, g_{cf}, \frac{l-1}{2} - \overline{p}(l+1)\right).$$

Furthermore, Poincaré duality is realised in the spaces on the right by $*_{\frac{l-1}{2}-\overline{p}(l+1)}$.

Proof outline

1. Prove isomorphism between cohomological spaces:

$$HI^j_{\overline{p}}(X) \cong IG^j_{(j+1-l+\overline{p}(l+1))}(CT(X))$$

2. The space on the right is isomorphic to a space of extended weighted L^2 harmonic forms for a fibred cusp metric:

$$IG^j_{(\frac{f}{2}-c+1)}(CT(X)) \cong \mathcal{H}^j_{ext}(\hat{X}, g_{fc}, c).$$

3. Because the link bundle is a product, g_{cf} is conformal to g_{fc}, so we get an identity:

$$\mathcal{H}^j_{ext}(\hat{X}, g_{fc}, c) = \mathcal{H}^j_{ext}(\hat{X}, g_{cf}, c + j - \frac{n}{2}),$$

where $n = \dim(X)$.

Topological surgery

The space $CT(X)$ appearing in the proof is called the *conifold transition* of X. Conifold transitions are a generalisation of topological surgery.

Recall topological surgery:

- $S^k \subset M^n$ has a trivial normal bundle,
- Remove a tubular neighborhood of S^k, $N(S^k) \cong S^k \times B(n-k)$
- Glue in $B^{k+1} \times S^{n-k-1}$ along the (identical) boundary.

Note that $B^{k+1} = C(S^k)$.

- Classical result: Topological signature is preserved by sugery.
- Surgery is used to obtain one important family of *mirror duals*, where the duals are referred to as conifold transitions of each other.

Conifold transitions

If X has a smooth singular stratum, Σ with product link bundle, L, that is,

$$X = \hat{X} \cup (\Sigma \times C(L)),$$

then we can generalise the idea of surgery to define a new pseudomanifold called the *conifold transition* of X:

$$CT(X) = \hat{X} \cup (C(\Sigma) \times L).$$

Theorem

(Banagl, H-) If X is a pseudomanifold with smooth singular stratum Σ and product link bundle, then

$$\sigma_{HI}(X) = \sigma_{IH}(CT(X)) = \sigma_{IH}(\overset{\circ}{X}) = \sigma_{HI}(\overset{\circ}{X}),$$

where $\overset{\circ}{X}$ is the one-point compactification of $\hat{X}$.

The spaces $IG_{\overline{p}}(CT(X))$

The second step of the proof generalises the Melrose result about extended L^2 harmonic forms for manifolds with infinite cylindrical ends.

$$IG^j_{(\frac{f}{2}-c+1)}(CT(X)) \cong \mathcal{H}^j_{ext}(\hat{X}, g_{fc}, c).$$

Recall in the cylindrical case, extended harmonic forms were isomorphic to the space:

$$G^j(X,\Sigma) \cong \frac{H^j(X,\Sigma) \oplus H^j(X)}{\mathrm{Im}(H^j(X,\Sigma) \to H^j(X))}.$$

In the fibred cusp setting, this generalises to:

$$IG^j_{(c)}(CT(X)) := \frac{IH^j_{\underline{p}}(CT(X)) \oplus IH^j_{\overline{p}}(CT(X))}{\mathrm{Im}(IH^j_{\underline{p}}(CT(X)) \to IH^j_{\overline{p}}(CT(X))},$$

where $f - \underline{p}(s+1) = \frac{f}{2} - c$ and $f - \overline{p}(s+1) = \frac{f}{2} - c + 1$.

Future directions

- Generalise to geometrically flat link bundles. Note here we can't define $CT(X)$, so the proof needs to be done directly.
- Generalise to pseudomanifolds with more than two strata–a first example of an intersection space $I^{\bar{p}}(X)$ has been defined in the three stratum case, but perhaps the Hodge theorem suggests a better way to generalise.