TECHNISCHE
UNIVERSITÄT
WIEN

Entropy method for hypocoercive Fokker-Planck equations with linear drift

Anton ARNOLD

(with Jan Erb)

Banff, July 2014

degenerate Fokker-Planck equations with linear drift

evolution of probability density $f(x,t)$, $x \in \mathbb{R}^n$, $t > 0$:

$$\begin{aligned} f_t &= \operatorname{div}\Big(\mathbf{D}\nabla f + \mathbf{C}x\, f\Big) \\ f(x,0) &= f_0(x) \end{aligned} \qquad (1)$$

$\mathbf{D} \in \mathbb{R}^{n\times n}$ ... symmetric, const in x, degenerate

w.r.o.g. (via coordinate transformation, x–scaling):
let $\mathbf{D} = \operatorname{diag}(\underbrace{1, \dots 1}_{k}, \underbrace{0, \dots 0}_{n-k})$

$\mathbf{C} \in \mathbb{R}^{n\times n}$ ... const in x

degenerate Fokker-Planck equations with linear drift

evolution of probability density $f(x,t)$, $x \in \mathbb{R}^n$, $t > 0$:

$$\begin{aligned} f_t &= \operatorname{div}\Big(\mathbf{D}\nabla f + \mathbf{C}x\, f\Big) \\ f(x,0) &= f_0(x) \end{aligned} \qquad (1)$$

$\mathbf{D} \in \mathbb{R}^{n\times n}$ ... symmetric, const in x, degenerate

w.r.o.g. (via coordinate transformation, x–scaling):
let $\mathbf{D} = \operatorname{diag}(\underbrace{1, \dots 1}_{k}, \underbrace{0, \dots 0}_{n-k})$

$\mathbf{C} \in \mathbb{R}^{n\times n}$ ... const in x

goals: existence & uniqueness of steady state $f_\infty(x)$;
convergence $f(t) \stackrel{t\to\infty}{\longrightarrow} f_\infty$ with sharp rates;
complete theory for the equation class (1)

hypocoercive example

kinetic Fokker-Planck equation for $f(x, v, t)$, $x, v \in \mathbb{R}^n$:

$$f_t + \underbrace{v \cdot \nabla_x f}_{\text{free transport}} - \underbrace{\nabla_x V \cdot \nabla_v f}_{\text{influence of potential } V(x,t)} = \underbrace{\sigma \Delta_v f}_{\text{diffusion, } \sigma>0} + \underbrace{\nu \operatorname{div}_v(vf)}_{\text{friction, } \nu>0}$$

steady state: $f_\infty(x, v) = c\, e^{-\frac{\nu}{\sigma}\left[\frac{|v|^2}{2} + V(x)\right]}$
$V(x)$... given *confinement potential*

hypocoercive example

kinetic Fokker-Planck equation for $f(x,v,t)\,,\ x,v\in\mathbb{R}^n$:

$$f_t + \underbrace{v\cdot\nabla_x f}_{\text{free transport}} - \underbrace{\nabla_x V\cdot\nabla_v f}_{\text{influence of potential } V(x,t)} = \underbrace{\sigma\Delta_v f}_{\text{diffusion, } \sigma>0} + \nu\,\underbrace{\mathrm{div}_v(vf)}_{\text{friction, } \nu>0}$$

steady state: $f_\infty(x,v) = c\, e^{-\frac{\nu}{\sigma}\left[\frac{|v|^2}{2}+V(x)\right]}$

$V(x)$... given *confinement potential*

rewritten:

$$f_t = \mathrm{div}_{x,v}\Big[\underbrace{\begin{pmatrix}0 & 0\\ 0 & \sigma\,\mathbf{I}\end{pmatrix}}_{=:\mathbf{D}\ldots\text{diffusion}}\nabla_{x,v}f + \underbrace{\begin{pmatrix}-v\\ \nabla_x V+\nu\,v\end{pmatrix}}_{\text{drift}} f\Big]$$

Outline:

1. review of standard entropy method for non-degenerate Fokker-Planck equations
2. hypocoercivity, prototypic examples
3. decay of modified "entropy dissipation" functional
4. regularization of semigroup $\rightarrow$ entropy decay
5. sharp decay rates

review of entropy method: linear symmetric Fokker-Planck equations

evolution of probability density $f(x,t)$, $x \in \mathbb{R}^n$, $t > 0$:

$$
\begin{aligned}
f_t &= \operatorname{div}\Big(\mathbf{D}(x) \cdot [\nabla A(x) f + \nabla f]\Big) =: Lf \\
f(x,0) &= f_0(x); \quad f_0 \in L^1_+(\mathbb{R}^n),\ \int_{\mathbb{R}^n} f_0 \, dx = 1 \quad \Rightarrow \quad f(x,t) \geq 0
\end{aligned}
$$

review of entropy method: linear symmetric Fokker-Planck equations

evolution of probability density $f(x,t)$, $x \in \mathbb{R}^n$, $t > 0$:

$$f_t = \operatorname{div}\Big(\mathbf{D}(x) \cdot [\nabla A(x) f + \nabla f]\Big) =: Lf$$

$$f(x,0) = f_0(x); \quad f_0 \in L^1_+(\mathbb{R}^n), \int_{\mathbb{R}^n} f_0 \, dx = 1 \quad \Rightarrow \quad f(x,t) \geq 0$$

$$f_\infty(x) = e^{-A(x)} \quad \ldots \text{ (unique) normalized steady state}$$

$$Lf = \operatorname{div}\Big(f_\infty \mathbf{D}(x) \nabla \frac{f}{f_\infty}\Big) \quad \ldots \text{ symmetric in } L^2(\mathbb{R}^n, f_\infty^{-1})$$

$\mathbf{D}(x) > 0$ ... positive definite matrix $\forall x \in \mathbb{R}^n$

$A(x)$ ... scalar confinement potential, i.e. $A(x) \to \infty$ as $|x| \to \infty$;
idea : $A(x) \gtrsim c|x|^2$

admissible relative entropies (for entropy method)

$\psi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ ... entropy generators

$$\psi \geq 0, \quad \psi(1) = 0, \quad \psi'' > 0, \quad (\psi''')^2 \leq \tfrac{1}{2}\psi''\psi^{\mathrm{IV}}$$

admissible relative entropies (for entropy method)

$\psi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ ... entropy generators

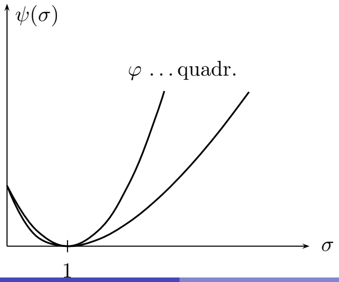
$\psi \geq 0, \quad \psi(1) = 0, \quad \psi'' > 0, \quad (\psi''')^2 \leq \frac{1}{2}\psi''\psi^{\mathrm{IV}}$

examples
1) $\psi_1(\sigma) = \sigma \ln \sigma - \sigma + 1$
2) $\psi_p(\sigma) = \sigma^p - 1 - p(\sigma - 1), \quad 1 < p \leq 2$

$$e_\psi(f_1|f_2) := \int_{\mathbb{R}^n} \psi\left(\frac{f_1}{f_2}\right) f_2 \, \mathrm{d}x \geq 0$$

... relative entropy

for $f_{1,2}$ ... probability densities



goals of the entropy method

- prove convergence $f(t) \stackrel{t\to\infty}{\longrightarrow} f_\infty$
- (possibly) with sharp exponential rate

goals of the entropy method

- prove convergence $f(t) \stackrel{t\to\infty}{\longrightarrow} f_\infty$
- (possibly) with sharp exponential rate

- convergence in relative entropy $\Rightarrow$ L^1–convergence by (generalized) Csiszár-Kullback inequality:

$$\|f_1 - f_2\|^2_{L^1(\mathbb{R}^n)} \le \frac{2}{\psi''(1)}\, e_\psi(f_1|f_2)$$

entropy dissipation

Lemma 1

Let $f(t)$ solve Fokker-Planck equation $f_t = div\Big(\mathbf{D}(x)\cdot[\nabla A(x)f + \nabla f]\Big)$

$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}t} e_\psi(f(t)|f_\infty) \quad = \quad -\int_{\mathbb{R}^n} \psi''\left(\frac{f(t)}{f_\infty}\right)\nabla^\top \frac{f(t)}{f_\infty}\cdot \mathbf{D}(x)\cdot \nabla \frac{f(t)}{f_\infty}\, f_\infty \,\mathrm{d}x$$

$$=: \quad -I_\psi(f(t)|f_\infty) \le 0 \text{ ... (negative) Fisher information}$$

exponential decay of entropy dissipation for $\mathbf{D} \equiv$ const.

$\mathbf{D}$ = const. in x, $\quad f_\infty(x) = e^{-A(x)}$

Theorem 1

Let $I_\psi(f_0|f_\infty) < \infty$. Let $\mathbf{D}, A$ satisfy a

$$\boxed{\textit{Bakry - Emery condition} \quad \frac{\partial^2 A(x)}{\partial x^2} \geq \underbrace{\lambda_1}_{>0} \mathbf{D}^{-1}} \quad \forall x \in \mathbb{R}^n \qquad (2)$$

$$\Rightarrow I_\psi(f(t)|f_\infty) \leq e^{-2\lambda_1 t} I_\psi(f_0|f_\infty), \qquad t \geq 0$$

A ... uniformly convex if $\mathbf{D} = \mathbf{I}$

Ref's: [Bakry-Emery] 1984/85;
[Arnold-Markowich-Toscani-Unterreiter] Comm. PDE 2001

proof of Theorem 1 in 1D with $\mathbf{D} \equiv 1$ (BEC: $A''(x) \geq \lambda_1$)

$$I_\psi(t) = \int \psi''\Big(\frac{f(t)}{f_\infty}\Big)\Big(\underbrace{\partial_x \frac{f(t)}{f_\infty}}_{=:u}\Big)^2 f_\infty \,\mathrm{d}x \geq 0 \qquad ; \quad f_t = \Big(f_\infty \,\partial_x \underbrace{\frac{f}{f_\infty}}_{=:u}\Big)_x$$

proof of Theorem 1 in 1D with $\mathbf{D} \equiv 1$ (BEC: $A''(x) \geq \lambda_1$)

$$I_\psi(t) = \int \psi''\Big(\frac{f(t)}{f_\infty}\Big)\Big(\underbrace{\partial_x \frac{f(t)}{f_\infty}}_{=:u}\Big)^2 f_\infty \,\mathrm{d}x \geq 0 \qquad ; \quad f_t = \Big(f_\infty \underbrace{\partial_x \frac{f}{f_\infty}}_{=:u}\Big)_x$$

$$\frac{\mathrm{d}}{\mathrm{d}t} I = \int \psi'''\Big(\frac{f}{f_\infty}\Big) \underbrace{(f_\infty u)_x}_{=f_t} u^2 \,\mathrm{d}x + 2\int \psi''\Big(\frac{f}{f_\infty}\Big) u \underbrace{u_t f_\infty}_{=(u_x f_\infty)_x - A'' u f_\infty} \,\mathrm{d}x$$

proof of Theorem 1 in 1D with $\mathbf{D} \equiv 1$ (BEC: $A''(x) \geq \lambda_1$)

$$I_\psi(t) = \int \psi''\Big(\frac{f(t)}{f_\infty}\Big)\Big(\underbrace{\partial_x \frac{f(t)}{f_\infty}}_{=:u}\Big)^2 f_\infty \,\mathrm{d}x \geq 0 \qquad ; \quad f_t = \Big(f_\infty \underbrace{\partial_x \frac{f}{f_\infty}}_{=:u}\Big)_x$$

$$\frac{\mathrm{d}}{\mathrm{d}t} I = \int \psi'''\Big(\frac{f}{f_\infty}\Big) \underbrace{(f_\infty u)_x}_{=f_t} u^2 \,\mathrm{d}x + 2\int \psi''\Big(\frac{f}{f_\infty}\Big) u \underbrace{u_t f_\infty}_{=(u_x f_\infty)_x - A'' u f_\infty} \mathrm{d}x$$

$$\underset{\text{2 int. by parts}}{=} -\int \psi^{\mathrm{IV}}\Big(\frac{f}{f_\infty}\Big) u^4 f_\infty \,\mathrm{d}x - 2\int \psi'''\Big(\frac{f}{f_\infty}\Big) u_x u^2 f_\infty \,\mathrm{d}x$$

$$\underbrace{-2\int \psi''\Big(\frac{f}{f_\infty}\Big) A'' u^2 f_\infty \,\mathrm{d}x}_{\geq 0} - 2\int \psi'''\Big(\frac{f}{f_\infty}\Big) u^2 u_x f_\infty - \psi''\Big(\frac{f}{f_\infty}\Big) {u_x}^2 f_\infty \,\mathrm{d}x =$$

proof of Theorem 1 in 1D with $\mathbf{D} \equiv 1$ (BEC: $A''(x) \geq \lambda_1$)

$$\frac{\mathrm{d}}{\mathrm{d}t} I = -2 \underbrace{\int \psi''\left(\frac{f}{f_\infty}\right) A''(x) u^2 f_\infty \,\mathrm{d}x}_{\geq 0} - 2 \int \underbrace{\mathrm{Tr}\,(XY)}_{\geq 0} \underbrace{f_\infty}_{\geq 0} \,\mathrm{d}x$$

proof of Theorem 1 in 1D with $\mathbf{D} \equiv 1$ (BEC: $A''(x) \geq \lambda_1$)

$$\frac{\mathrm{d}}{\mathrm{d}t} I = -2 \underbrace{\int \psi''\Big(\frac{f}{f_\infty}\Big) A''(x) u^2 f_\infty \, \mathrm{d}x}_{\geq 0} - 2 \int \underbrace{\mathrm{Tr}\,(XY)}_{\geq 0} \underbrace{f_\infty}_{\geq 0} \, \mathrm{d}x$$

$$\overset{\text{BEC}}{\leq} \quad -2\lambda_1 \int \psi''\Big(\frac{f}{f_\infty}\Big) u^2 f_\infty \, \mathrm{d}x = -2\lambda_1 I(t)$$

proof of Theorem 1 in 1D with $\mathbf{D} \equiv 1$ (BEC: $A''(x) \geq \lambda_1$)

$$\frac{\mathrm{d}}{\mathrm{d}t} I = -2 \underbrace{\int \psi''\left(\frac{f}{f_\infty}\right) A''(x) u^2 f_\infty \, \mathrm{d}x}_{\geq 0} - 2 \int \underbrace{\mathrm{Tr}\,(XY)}_{\geq 0} \underbrace{f_\infty}_{\geq 0} \, \mathrm{d}x$$

$$\overset{\text{BEC}}{\leq} \quad -2\lambda_1 \int \psi''\left(\tfrac{f}{f_\infty}\right) u^2 f_\infty \, \mathrm{d}x = -2\lambda_1 I(t)$$

use

$$X := \begin{pmatrix} \psi'' & \psi''' \\ \psi''' & \frac{1}{2}\psi^{\mathrm{IV}} \end{pmatrix} \left(\frac{f}{f_\infty}\right) \geq 0 \quad ; Y := \begin{pmatrix} u_x^2 & u^2 u_x \\ u^2 u_x & u^4 \end{pmatrix} \geq 0$$

with $\det X = \frac{1}{2}\psi''\psi^{\mathrm{IV}} - (\psi''')^2 \geq 0$ (for admissible entropies) .

proof of Theorem 1 in 1D with $\mathbf{D} \equiv 1$ (BEC: $A''(x) \geq \lambda_1$)

$$\frac{\mathrm{d}}{\mathrm{d}t} I = \underbrace{-2 \int \psi''\left(\frac{f}{f_\infty}\right) A''(x) u^2 f_\infty \, \mathrm{d}x}_{\geq 0} - 2 \int \underbrace{\mathrm{Tr}\,(XY)}_{\geq 0} \underbrace{f_\infty}_{\geq 0} \, \mathrm{d}x$$

$$\stackrel{\text{BEC}}{\leq} \quad -2\lambda_1 \int \psi''\left(\frac{f}{f_\infty}\right) u^2 f_\infty \, \mathrm{d}x = -2\lambda_1 I(t)$$

use

$$X := \begin{pmatrix} \psi'' & \psi''' \\ \psi''' & \frac{1}{2}\psi^{\mathrm{IV}} \end{pmatrix} \left(\frac{f}{f_\infty}\right) \geq 0 \quad ; Y := \begin{pmatrix} u_x^2 & u^2 u_x \\ u^2 u_x & u^4 \end{pmatrix} \geq 0$$

with $\det X = \frac{1}{2}\psi''\psi^{\mathrm{IV}} - (\psi''')^2 \geq 0$ (for admissible entropies).

$$\Rightarrow I_\psi(f(t)|f_\infty) \leq \mathrm{e}^{-2\lambda_1 t} I_\psi(f_0|f_\infty)$$

□

exponential decay of relative entropy for $\mathbf{D} \equiv$ const.

Theorem 2

Let $\mathbf{D}$, A *satisfy BEC* $\quad \frac{\partial^2 A(x)}{\partial x^2} \geq \lambda_1 \mathbf{D}^{-1} \Rightarrow$

$$e_\psi(f(t)|f_\infty) \leq e^{-2\lambda_1 t}\, e_\psi(f_0|f_\infty), \qquad t \geq 0$$

exponential decay of relative entropy for $\mathbf{D} \equiv$ const.

Theorem 2

Let $\mathbf{D}$, *A satisfy BEC* $\quad \frac{\partial^2 A(x)}{\partial x^2} \geq \lambda_1 \mathbf{D}^{-1} \Rightarrow$

$$e_\psi(f(t)|f_\infty) \leq e^{-2\lambda_1 t} e_\psi(f_0|f_\infty), \qquad t \geq 0$$

Proof: from proof of Theorem 1 :

$$\frac{\mathrm{d}}{\mathrm{d}t} I(t) \leq -2\lambda_1 \underbrace{I(t)}_{=-e'(t)} \qquad \Big| \int_t^\infty \ldots \, \mathrm{d}t$$

Since $I(t)$, $e(t) \xrightarrow{t\to\infty} 0$:

$$\frac{\mathrm{d}}{\mathrm{d}t} e(t) \leq -2\lambda_1 e(t) \qquad (3)$$

(+ density argument) □

(hypo)coercivity 1

example 1: standard Fokker-Planck equation on $\mathbb{R}^n$:

$$f_t = \operatorname{div}\Big(\nabla f + x\, f\Big) =: Lf \quad \ldots \text{ symmetric on } \; H := L^2(f_\infty^{-1})$$

$$f_\infty(x) = c e^{-\frac{|x|^2}{2}}, \qquad \ker L = \operatorname{span}(f_\infty)$$

L is dissipative, i.e. $\langle Lf, f\rangle_H \le 0 \quad \forall f \in \mathcal{D}(L)$

$-L$ is coercive (has a spectral gap), in the sense:

$$\langle -Lf, f\rangle_H \ge \|f\|^2_{L^2(f_\infty^{-1})} \quad \forall\, f \in \{f_\infty\}^\perp$$

(hypo)coercivity 2

example 2:

$$f_t \;=\; \operatorname{div}\Big(\mathbf{D}\nabla f + \mathbf{C}x\, f\Big) =: Lf \qquad (4)$$

with degenerate $\mathbf{D}$ is degenerate parabolic;
(symmetric part of) $-L$ is not coercive.

(hypo)coercivity 2

example 2:

$$f_t \quad = \quad \operatorname{div}\Big(\mathbf{D}\nabla f + \mathbf{C}x\, f\Big) =: Lf \tag{4}$$

with degenerate $\mathbf{D}$ is degenerate parabolic;
(symmetric part of) $-L$ is not coercive.

Definition 3 (Villani 2009)

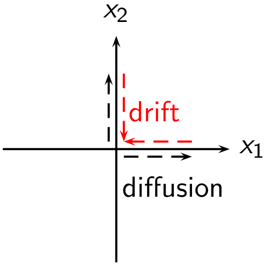
Consider L on Hilbert space H with $\mathcal{K} = \ker L$; let $\tilde{H} \hookrightarrow \mathcal{K}^\perp$ (densely) (e.g. H ... weighted L^2, $\tilde{H}$ ... weighted H^1).
$-L$ is called hypocoercive on $\tilde{H}$ if $\exists\, \lambda > 0,\ c > 0$:

$$\|e^{Lt} f\|_{\tilde{H}} \leq c\, e^{-\lambda t} \|f\|_{\tilde{H}} \qquad \forall f \in \tilde{H}$$

problem 1 (cp. standard entropy method): steady state

standard Fokker-Planck equation $f_t = \text{div}(\nabla f + x\, f)$:
unique steady state $f_\infty(x) = c\, \mathrm{e}^{-|x|^2/2}$ as a balance of drift & diffusion;
sharp decay rate $= 1$

$n = 2$:

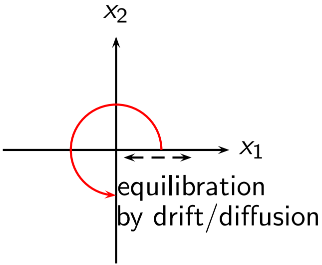


2 degenerate prototypes

prototype (a): degenerate diffusion (1D Fokker-Planck) + rotation

$$f_t = \operatorname{div}\Big[\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{=\mathbf{D}} \nabla f + \underbrace{\begin{pmatrix} x_1 - \omega x_2 \\ \omega x_1 \end{pmatrix}}_{=\mathbf{C}x} f\Big]$$

$f_\infty(x) = c\, e^{-|x|^2/2} \qquad \forall\, \omega \in \mathbb{R}$ (unique for $\omega \neq 0$);
sharp decay rate $= \frac{1}{2}$ ($= \min \Re\, \lambda_C$) for fast enough rotation (i.e. $|\omega| > \frac{1}{2}$)

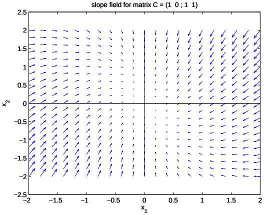


2 degenerate prototypes

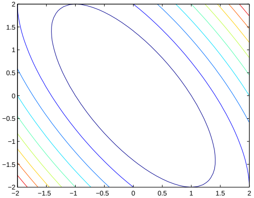
prototype (b): degenerate diffusion – not aligned with drift characteristics

$$f_t \;=\; \operatorname{div}\Big[\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{=\mathbf{D}} \nabla f + \underbrace{\begin{pmatrix} x_1 \\ x_1 + x_2 \end{pmatrix}}_{=\mathbf{C}x} f\Big]$$

$f_\infty(x) = c\, e^{-(x_1^2 + 2x_1x_2 + 2x_2^2)}$; (sharp) decay rate $= 1 - \varepsilon$ ($< \min \Re\, \lambda_C$)



characteristics of drift: $x_t = -\mathbf{C}\, x$



contours of steady state potential $-\ln f_\infty$

coefficients **C**, **D** in Fokker-Planck equation

$$f_t = \operatorname{div}\Big(\mathbf{D}\,\nabla f + \mathbf{C}\,x\,f\Big) =: Lf$$

Condition A: No (nontrivial) subspace of $\ker \mathbf{D}$ is invariant under $\mathbf{C}^\top$.
(equivalent: No eigenvector v of $\mathbf{C}^\top$ satisfies $\mathbf{D}\,v = 0$. L hypoelliptic.)

Proposition 1

Let Condition A hold.

a) *Let* $f_0 \in L^1(\mathbb{R}^d) \quad \Rightarrow \quad f \in C^\infty(\mathbb{R}^n \times \mathbb{R}^+)$. *[Hörmander 1969]*

b) *Let* $f_0 \in L^1_+(\mathbb{R}^d) \quad \Rightarrow \quad f(x,t) > 0,\ \forall t > 0$. *(Green's fct* > 0*)*

coefficients **C**, **D** in Fokker-Planck equation

$$f_t = \operatorname{div}\Big(\mathbf{D}\,\nabla f + \mathbf{C}\,x\,f\Big) =: Lf$$

<u>Condition A:</u> No (nontrivial) subspace of $\ker \mathbf{D}$ is invariant under $\mathbf{C}^\top$.
(equivalent: No eigenvector v of $\mathbf{C}^\top$ satisfies $\mathbf{D}\,v = 0$. L hypoelliptic.)

Proposition 1

Let Condition A hold.

a) *Let* $f_0 \in L^1(\mathbb{R}^d) \quad \Rightarrow \quad f \in C^\infty(\mathbb{R}^n \times \mathbb{R}^+)$. *[Hörmander 1969]*

b) *Let* $f_0 \in L^1_+(\mathbb{R}^d) \quad \Rightarrow \quad f(x,t) > 0,\ \forall t > 0$. *(Green's fct* > 0*)*

<u>Condition B:</u> Condition A + let **C** be positively stable (i.e. $\Re\,\lambda_C > 0$)
$\rightarrow \exists$ confinement potential; drift towards $x = 0$.

steady state

$$f_t = \operatorname{div}\Big(\mathbf{D}\nabla f + \mathbf{C}x\, f\Big) \tag{5}$$

Theorem 4

(5) *has a unique (normalized) steady state* $f_\infty \in L^1(\mathbb{R}^n)$ *iff Condition B holds.*

Then: $f_\infty(x) = c_K e^{-\frac{x^\top \mathbf{K}^{-1} x}{2}}$ ... *non-isotropic Gaussian*

$0 < \mathbf{K} \in \mathbb{R}^{n\times n}$ ... *unique solution of* $2\mathbf{D} = \mathbf{C}\mathbf{K} + \mathbf{K}\mathbf{C}^\top$

(continuous Lyapunov equation)

steady state

$$f_t = \operatorname{div}\Big(\mathbf{D}\nabla f + \mathbf{C}x\, f\Big) \tag{5}$$

Theorem 4

(5) *has a unique (normalized) steady state* $f_\infty \in L^1(\mathbb{R}^n)$ *iff Condition B holds.*

Then: $f_\infty(x) = c_K e^{-\frac{x^\top \mathbf{K}^{-1} x}{2}}$ ... *non-isotropic Gaussian*

$0 < \mathbf{K} \in \mathbb{R}^{n\times n}$ ... *unique solution of* $2\mathbf{D} = \mathbf{C}\mathbf{K} + \mathbf{K}\mathbf{C}^\top$

(continuous Lyapunov equation)

proof - idea:

Fourier transformed equation for $\widehat{f}(\xi, t)$:

$$\widehat{f}_t = -(\xi^\top \mathbf{D}\xi)\widehat{f} - \left(\mathbf{C}^\top \xi\right)\cdot \nabla_\xi \widehat{f}, \qquad \text{ansatz: } \widehat{f}_\infty = c\, e^{-\frac{\xi^\top \mathbf{K}\xi}{2}}$$

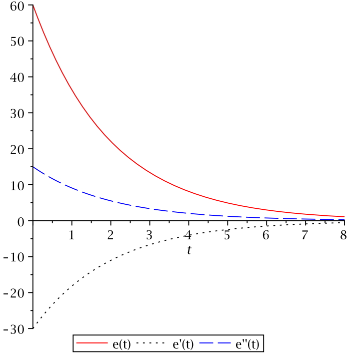
$\mathbf{D} \geq 0 \;\Rightarrow\; \exists\,!\; \mathbf{K} \geq 0\,;\quad \mathbf{K} > 0$ from Condition B.

decomposition of the generator L:

$$
\begin{aligned}
\partial_t f &= Lf = \operatorname{div}\Big(\mathbf{D}\nabla f + \mathbf{C}x\, f\Big) \quad \text{in } L^2(f_\infty^{-1}), \qquad (6)\\
L &= L^s + L^{as} \quad \text{with} \quad L^s f_\infty = L^{as} f_\infty = 0,\\
L^s f &= \operatorname{div}\Big(\mathbf{D}\Big(\nabla \frac{f}{f_\infty}\Big) f_\infty\Big) \quad \ldots \text{ like in non-degenerate case},\\
L^{as} f &= \operatorname{div}\Big(\mathbf{R}\Big(\nabla \frac{f}{f_\infty}\Big) f_\infty\Big),\\
\mathbf{R} &:= -\mathbf{R}^\top = \frac{1}{2}(\mathbf{CK} - \mathbf{KC}^\top) \neq 0 \quad \rightarrow \quad \text{(6) is non-symmetric.}
\end{aligned}
$$

problem 2 (cp. standard entropy method): entropy decay

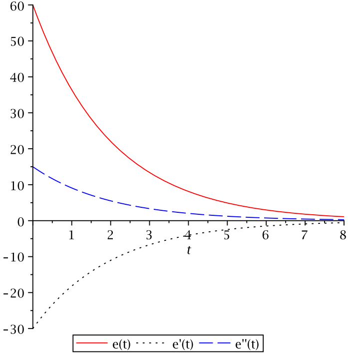
decay of quadratic entropy $e_2(t) = \|f(t) - f_\infty\|^2_{L^2}$:

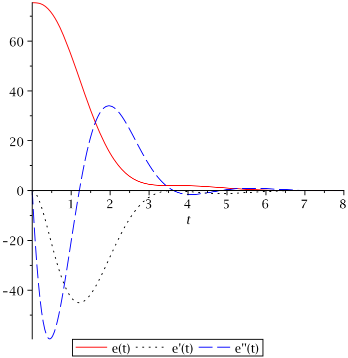


standard Fokker-Planck equation:
non-degenerate $\rightarrow$ $e(t)$ is convex;
entropy dissip. $e'(t) < 0 \ \forall \ f \neq f_\infty$;
$e' \leq -\mu e$ possible (with $\mu > 0$)

problem 2 (cp. standard entropy method): entropy decay

decay of quadratic entropy $e_2(t) = \|f(t) - f_\infty\|_{L^2}^2$:





standard Fokker-Planck equation:
non-degenerate → $e(t)$ is convex;
entropy dissip. $e'(t) < 0 \ \forall\, f \neq f_\infty$;
$e' \leq -\mu e$ possible (with $\mu > 0$)

degenerate prototype ex. (a):
→ $e(t)$ is not convex;
$e'(t) = 0$ for some $f \neq f_\infty$;
$e' \leq -\mu e$ wrong (in general)

modified entropy method for degenerate FP equation

$e'(t) = 0$ for some $f \neq f_\infty \quad \Rightarrow \quad$ entropy dissipation:

$$\frac{\mathrm{d}}{\mathrm{d}t} e_\psi = -\int_{\mathbb{R}^n} \psi''\left(\frac{f}{f_\infty}\right) \nabla^\top \frac{f}{f_\infty} \cdot \underbrace{\mathbf{D}}_{\geq 0} \cdot \nabla \frac{f}{f_\infty} f_\infty \, \mathrm{d}x =: -I_\psi(f) \leq 0$$

is "useless" as Lyapunov functional.

modified entropy method for degenerate FP equation

$e'(t) = 0$ for some $f \neq f_\infty \quad \Rightarrow \quad$ entropy dissipation:

$$\frac{\mathrm{d}}{\mathrm{d}t} e_\psi = - \int_{\mathbb{R}^n} \psi''\left(\frac{f}{f_\infty}\right) \nabla^\top \frac{f}{f_\infty} \cdot \underbrace{\mathbf{D}}_{\geq 0} \cdot \nabla \frac{f}{f_\infty} f_\infty \, \mathrm{d}x =: -I_\psi(f) \leq 0$$

is "useless" as Lyapunov functional.

$\Rightarrow$ define modified "entropy dissipation" as auxiliary functional:

$$S_\psi(f) := \int_{\mathbb{R}^n} \psi''\left(\frac{f}{f_\infty}\right) \nabla^\top \frac{f}{f_\infty} \cdot \underbrace{\mathbf{P}}_{>0} \cdot \nabla \frac{f}{f_\infty} f_\infty \, \mathrm{d}x \geq 0$$

goal: estimate between $S(f(t))$, $\frac{\mathrm{d}}{\mathrm{d}t} S(f(t))$ for "good" choice of $\mathbf{P} > 0$.
Then:

$$\mathbf{P} \geq c_P \, \mathbf{D} \quad \Rightarrow \quad S_\psi(f) \geq c_P \, I_\psi(f) \quad \searrow 0$$

modified "entropy dissipation" $S_\psi(f)$: choice of $\mathbf{P}$

Lemma 2

Let $\mu := \min\{\Re\,\lambda_C\}$ *(*> 0 *since* $\mathbf{C}$ *is positively stable);* $\mathbf{Q} := \mathbf{K}\mathbf{C}^\top\mathbf{K}^{-1}$ *.*

1. *If all* $\lambda_C^{\min} := \{\lambda \in \sigma(\mathbf{C}) \,|\, \Re\lambda = \mu\}$ *are non-defective (i.e. geometric = algebraic multiplicity)*
   $\Rightarrow \quad \exists\, \mathbf{P} \in \mathbb{R}^{n\times n},\ \mathbf{P} > 0 \ : \quad \mathbf{Q}\mathbf{P} + \mathbf{P}\mathbf{Q}^\top \geq 2\mu\mathbf{P}$ *.*

modified "entropy dissipation" $S_\psi(f)$: choice of $\mathbf{P}$

Lemma 2

Let $\mu := \min\{\Re\,\lambda_C\}$ *(*> 0 *since* $\mathbf{C}$ *is positively stable);* $\mathbf{Q} := \mathbf{K}\mathbf{C}^\top\mathbf{K}^{-1}$ *.*

1. *If all* $\lambda_C^{\min} := \{\lambda \in \sigma(\mathbf{C}) \,|\, \Re\lambda = \mu\}$ *are non-defective* *(i.e. geometric = algebraic multiplicity)*
   $\Rightarrow \quad \exists\, \mathbf{P} \in \mathbb{R}^{n\times n},\ \mathbf{P} > 0\ : \quad \mathbf{QP} + \mathbf{PQ}^\top \geq 2\mu\mathbf{P}\,.$
2. *If (at least) one* $\lambda_C^{\min}$ *is defective* $\Rightarrow$
   $\forall\,\varepsilon > 0 \quad \exists\,\mathbf{P} = \mathbf{P}(\varepsilon) > 0\ : \quad \mathbf{QP} + \mathbf{PQ}^\top \geq 2(\mu - \varepsilon)\mathbf{P}\,.$

modified "entropy dissipation" $S_\psi(f)$: choice of **P**

Lemma 2

Let $\mu := \min\{\Re\,\lambda_C\}$ *(*> 0 *since* **C** *is positively stable);* $\mathbf{Q} := \mathbf{K}\mathbf{C}^\top\mathbf{K}^{-1}$ *.*

1. *If all* $\lambda_C^{\min} := \{\lambda \in \sigma(\mathbf{C}) \,|\, \Re\lambda = \mu\}$ *are non-defective (i.e. geometric = algebraic multiplicity)*
$\Rightarrow \quad \exists\, \mathbf{P} \in \mathbb{R}^{n\times n},\ \mathbf{P} > 0 \ : \quad \mathbf{QP} + \mathbf{PQ}^\top \geq 2\mu\mathbf{P}\,.$
2. *If (at least) one* $\lambda_C^{\min}$ *is defective* $\Rightarrow$
$\forall\, \varepsilon > 0 \quad \exists\, \mathbf{P} = \mathbf{P}(\varepsilon) > 0 \ : \quad \mathbf{QP} + \mathbf{PQ}^\top \geq 2(\mu - \varepsilon)\mathbf{P}\,.$

Proof: **P** can be constructed explicitly; e.g. for **C** non-defective / diagonalizable:

$$\mathbf{P} := \sum_{j=1}^{n} z_j \otimes \bar{z}_j^\top\ ; \qquad z_j \ldots \text{eigenvectors of } \mathbf{Q}$$

- **P** not unique; but decay are rates independent of **P**

exponential decay of auxiliary functional $S_\psi(f)$

Proposition 2

$\mu := \min\{\Re\,\lambda_C\}$. *Let* f_0 *satisfy:*

$$\int \psi''\left(\frac{f_0}{f_\infty}\right)\left|\nabla\frac{f_0}{f_\infty}\right|^2 f_\infty\,\mathrm{dx} < \infty \quad (\sim \textit{ weighted } H^1\textit{–seminorm})$$

1. *If all* $\lambda_C^{\min}$ *are non-defective* $\Rightarrow \quad S(f(t)) \le e^{-2\mu t} S(f_0)\,, \quad t \ge 0;$

exponential decay of auxiliary functional $S_\psi(f)$

Proposition 2

$\mu := \min\{\Re\,\lambda_C\}$. *Let* f_0 *satisfy:*

$$\int \psi''\left(\frac{f_0}{f_\infty}\right)\left|\nabla\frac{f_0}{f_\infty}\right|^2 f_\infty\,\mathrm{d}x < \infty \quad (\sim \text{ weighted } H^1\text{–seminorm})$$

1. *If all* $\lambda_C^{\min}$ *are non-defective* $\Rightarrow \quad S(f(t)) \le e^{-2\mu t}S(f_0)\,, \quad t \ge 0;$
2. *If one* $\lambda_C^{\min}$ *is defective* $\Rightarrow \quad S(f(t),\varepsilon) \le e^{-2(\mu-\varepsilon)t}S(f_0,\varepsilon)\,, \quad t \ge 0.$

exponential decay of auxiliary functional $S_\psi(f)$

Proposition 2

$\mu := \min\{\Re \lambda_C\}$. *Let* f_0 *satisfy:*

$$\int \psi''\left(\frac{f_0}{f_\infty}\right)\left|\nabla \frac{f_0}{f_\infty}\right|^2 f_\infty \, \mathrm{d}x < \infty \quad (\sim \textit{ weighted } H^1\textit{–seminorm})$$

1. *If all* $\lambda_C^{\min}$ *are non-defective* $\Rightarrow \quad S(f(t)) \le e^{-2\mu t} S(f_0)\,, \quad t \ge 0;$
2. *If one* $\lambda_C^{\min}$ *is defective* $\Rightarrow \quad S(f(t), \varepsilon) \le e^{-2(\mu-\varepsilon)t} S(f_0, \varepsilon)\,, \quad t \ge 0.$

notation:

$$u(x,t) := \nabla \frac{f(x,t)}{f_\infty(x)}\,;$$

$$f_\infty(x) = c_K \mathrm{e}^{-\frac{x^\top \mathbf{K}^{-1} x}{2}} = c_K \mathrm{e}^{-V(x)}$$

Proof of Proposition 2 – modified entropy method

$$\frac{\mathrm{d}}{\mathrm{d}t}S(f(t)) = -\int \psi''\Big(\frac{f}{f_\infty}\Big) u^\top \underbrace{\left[(\mathbf{D}-\mathbf{R})\frac{\partial^2 V}{\partial x^2}\mathbf{P} + \mathbf{P}\frac{\partial^2 V}{\partial x^2}(\mathbf{D}+\mathbf{R})\right]}_{=\mathbf{QP}+\mathbf{PQ}^\top \geq 2\mu\mathbf{P}\ \dots\ \text{replaces BEC}} u f_\infty \,\mathrm{d}x$$

$$-2\int \underbrace{\mathrm{Tr}\,(XY)}_{\geq 0} f_\infty \,\mathrm{d}x \leq -2\mu\, S(f(t))$$

Proof of Proposition 2 – modified entropy method

$$\frac{\mathrm{d}}{\mathrm{d}t}S(f(t)) = -\int \psi''\Big(\frac{f}{f_\infty}\Big)u^\top \underbrace{\left[(\mathbf{D}-\mathbf{R})\frac{\partial^2 V}{\partial x^2}\mathbf{P} + \mathbf{P}\frac{\partial^2 V}{\partial x^2}(\mathbf{D}+\mathbf{R})\right]}_{=\mathbf{QP}+\mathbf{PQ}^\top \geq 2\mu\mathbf{P}\ \text{... replaces BEC}} uf_\infty \,\mathrm{d}x$$

$$-2\int \underbrace{\mathrm{Tr}\,(XY)}_{\geq 0} f_\infty \,\mathrm{d}x \leq -2\mu\, S(f(t))$$

use $X := \begin{pmatrix} \psi'' & \psi''' \\ \psi''' & \frac{1}{2}\psi^{\mathrm{IV}} \end{pmatrix}\Big(\frac{f}{f_\infty}\Big) \geq 0$;

$\det X = \frac{1}{2}\psi''\psi^{\mathrm{IV}} - (\psi'')^2 \geq 0$ (for admissible rel. entropies)

Proof of Proposition 2 – modified entropy method

$$\frac{\mathrm{d}}{\mathrm{d}t} S(f(t)) = -\int \psi''\Big(\frac{f}{f_\infty}\Big) u^\top \underbrace{\Big[(\mathbf{D}-\mathbf{R})\frac{\partial^2 V}{\partial x^2}\mathbf{P} + \mathbf{P}\frac{\partial^2 V}{\partial x^2}(\mathbf{D}+\mathbf{R})\Big]}_{=\mathbf{QP}+\mathbf{PQ}^\top \geq 2\mu \mathbf{P} \text{ ... replaces BEC}} u f_\infty \, \mathrm{d}x$$

$$-2\int \underbrace{\mathrm{Tr}\,(XY)}_{\geq 0} f_\infty \, \mathrm{d}x \leq -2\mu\, S(f(t))$$

use $\quad X := \begin{pmatrix} \psi'' & \psi''' \\ \psi''' & \frac{1}{2}\psi^{\mathrm{IV}} \end{pmatrix} \Big(\frac{f}{f_\infty}\Big) \geq 0 \quad ;$

$$\det X = \frac{1}{2}\psi''\psi^{\mathrm{IV}} - (\psi'')^2 \geq 0 \quad \text{(for admissible rel. entropies)}$$

$$Y := \begin{pmatrix} \mathrm{Tr}\,(\mathbf{D}\frac{\partial u}{\partial x}\mathbf{P}\frac{\partial u}{\partial x}) & u^\top \mathbf{D}\frac{\partial u}{\partial x}\mathbf{P}u \\ u^\top \mathbf{D}\frac{\partial u}{\partial x}\mathbf{P}u & (u^\top \mathbf{P}u)\,(u^\top \mathbf{D}u) \end{pmatrix} \geq 0\,, \quad \text{with}$$

Cauchy-Schwarz for $(u^\top \mathbf{D}\frac{\partial u}{\partial x}\mathbf{P}u)^2 = \mathrm{Tr}\,(\sqrt{\mathbf{P}}uu^\top\sqrt{\mathbf{D}} \cdot \sqrt{\mathbf{D}}\frac{\partial u}{\partial x}\sqrt{\mathbf{P}})^2$

exponential decay of relative entropy

Theorem 5

Let f_0 satisfy:

$$\int \psi''\left(\frac{f_0}{f_\infty}\right) |u|^2 \, f_\infty \, \mathrm{d}x < \infty \, .$$

$$\Rightarrow \quad \boxed{e(f(t)|f_\infty) \le c\, S(f(t)) \le c\, e^{-2\mu t} S(f_0)\,, \quad t \ge 0}$$

(reduced rate for a defective $\lambda_C^{\min}$: $2(\mu - \varepsilon)$)

exponential decay of relative entropy

Theorem 5

Let f_0 satisfy:

$$\int \psi''\left(\frac{f_0}{f_\infty}\right) |u|^2 \; f_\infty \, \mathrm{d}x < \infty \, .$$

$$\Rightarrow \quad \boxed{e(f(t)|f_\infty) \leq c\, S(f(t)) \leq c\, e^{-2\mu t} S(f_0)\,, \quad t \geq 0}$$

(reduced rate for a defective $\lambda_C^{\min}$: $2(\mu - \varepsilon)$)

Proof: Consider non-degenerate (auxiliary) symmetric FP equation:

$$g_t = \mathrm{div}\left(\mathbf{P}(\nabla \frac{g}{f_\infty}) f_\infty\right) \; ; \quad g_\infty = f_\infty = c_K \, e^{-V(x)} \qquad (7)$$

It satisfies the Bakry-Emery condition $\frac{\partial^2 V}{\partial x^2} = \mathbf{K}^{-1} \geq \lambda_P \mathbf{P}^{-1}$.

$$\Rightarrow \quad \text{convex Sobolev inequality:} \quad e_\psi(g|f_\infty) \leq \frac{1}{2\lambda_P} S_\psi(g) \quad \forall \, g$$

Remark: $S_\psi(g)$ is the true entropy dissipation for (7) ! □

(parabolic) regularization of semigroup e^{Lt}

Proposition 3

Let $m \leq n - k$ ($k =$rank $\mathbf{D}$) be the minimum such that

$$\sum_{j=0}^{m} \mathbf{C}^j \mathbf{D} (\mathbf{C}^\top)^j \geq \kappa \, \mathbf{I} \quad \textit{for some } \kappa > 0 \, .$$

(Existence of m is equivalent to Condition A, i.e. hypoellipticity of L.)

$$\Rightarrow \quad S_\psi(f(t)) \leq c \, t^{-(2m+1)} \, e_\psi(f_0 | f_\infty) \, , \quad 0 < t \leq 1 \, .$$

(parabolic) regularization of semigroup e^{Lt}

Proposition 3

Let $m \leq n-k$ ($k =$rank $\mathbf{D}$) be the minimum such that

$$\sum_{j=0}^{m} \mathbf{C}^j \mathbf{D} (\mathbf{C}^\top)^j \geq \kappa \, \mathbf{I} \quad \text{for some } \kappa > 0 \, .$$

(Existence of m is equivalent to Condition A, i.e. hypoellipticity of L.)

$$\Rightarrow \quad S_\psi(f(t)) \leq c \, t^{-(2m+1)} \, e_\psi(f_0|f_\infty) \, , \quad 0 < t \leq 1 \, .$$

<u>Ref's:</u>
Prop. 3 is generalization to all admissible relative entropies of:
[Hérau] JFA 2007;
[Villani] book 2009 (only for quadratic & logarithmic entropies)

Proof of Proposition 3

Prove decay of the auxiliary functional $\mathcal{F}(t)$:

$$\mathcal{F}(t) := \underbrace{c_1}_{>0} e_\psi(f(t)|f_\infty) + \underbrace{\int \psi''\left(\frac{f}{f_\infty}\right) u^\top \tilde{\mathbf{P}}(t) u\, f_\infty \,\mathrm{d}x}_{\text{“similar” to } S_\psi(f(t))} \geq 0\,,$$

$\tilde{\mathbf{P}}(t)$ ... matrix polynomial in t of order $2m+1$ (coeff's depend on $\mathbf{D}$, $\mathbf{Q}$) with

$$\tilde{\mathbf{P}}(t) \geq c_2 t^{2m+1}\mathbf{I} \geq c_3 t^{2m+1}\mathbf{P} > 0\,; \quad \tilde{\mathbf{P}}(0) = 0\,.$$

Proof of Proposition 3

Prove decay of the auxiliary functional $\mathcal{F}(t)$:

$$\mathcal{F}(t) := \underbrace{c_1}_{>0} e_\psi(f(t)|f_\infty) + \underbrace{\int \psi''\left(\frac{f}{f_\infty}\right) u^\top \tilde{\mathbf{P}}(t) u \, f_\infty \, \mathrm{d}x}_{\text{"similar" to } S_\psi(f(t))} \geq 0 \,,$$

$\tilde{\mathbf{P}}(t)$ ... matrix polynomial in t of order $2m+1$ (coeff's depend on $\mathbf{D}$, $\mathbf{Q}$) with

$$\tilde{\mathbf{P}}(t) \geq c_2 t^{2m+1} \mathbf{I} \geq c_3 t^{2m+1} \mathbf{P} > 0 \,; \quad \tilde{\mathbf{P}}(0) = 0 \,.$$

- computations like in modified entropy method (for $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}$) $\quad \Rightarrow$

$$c_1 \, e_\psi(f_0|f_\infty) = \mathcal{F}(0) \geq \mathcal{F}(t) \geq c_3 t^{2m+1} S_\psi(f(t))$$

□

exp. decay of rel. entropy for $f_t = \text{div}(\mathbf{D}\nabla f + \mathbf{C}x\, f) =: Lf$

combination of regularization for initial time with Th.5 (entropy decay) $\Rightarrow$

Theorem 6 (Erb-Arnold 2014)

Let L satisfy Condition B; $\mu := \min\{\Re\,\lambda_C\}$. $\Rightarrow \exists\, c > 0$:

$$\boxed{e_\psi(f(t)|f_\infty) \le c\, e^{-2\mu t} e_\psi(f_0|f_\infty)\,, \quad t \ge 0}$$

(reduced rate for a defective $\lambda_C^{\min}$: $2(\mu - \varepsilon)$)

exp. decay of rel. entropy for $f_t = \mathrm{div}(\mathbf{D}\nabla f + \mathbf{C}x\, f) =: Lf$

combination of regularization for initial time with Th.5 (entropy decay) $\Rightarrow$

Theorem 6 (Erb-Arnold 2014)

Let L satisfy Condition B; $\mu := \min\{\Re\, \lambda_C\}$. $\Rightarrow \exists\, c > 0$:

$$\boxed{e_\psi(f(t)|f_\infty) \le c\, e^{-2\mu t} e_\psi(f_0|f_\infty)\,, \quad t \ge 0}$$

(reduced rate for a defective $\lambda_C^{\min}$: $2(\mu - \varepsilon)$)

Proof:

$$e(t) \overset{\text{CSI}}{\le} \frac{1}{2\lambda_P} S(f(t)) \overset{\text{decay}}{\le} \frac{1}{2\lambda_P} \mathrm{e}^{-2\mu(t-\delta)} S(f(\delta)) \overset{\text{regulariz.}}{\le} c(\delta)\, e^{-2\mu t} \mathrm{e}(0)$$

□

exp. decay of rel. entropy for $f_t = \text{div}(\mathbf{D}\nabla f + \mathbf{C}x\, f) =: Lf$

combination of regularization for initial time with Th.5 (entropy decay) $\Rightarrow$

Theorem 6 (Erb-Arnold 2014)

Let L satisfy Condition B; $\mu := \min\{\Re\, \lambda_C\}$. $\Rightarrow \exists\, c > 0$:

$$\boxed{e_\psi(f(t)|f_\infty) \le c\, e^{-2\mu t} e_\psi(f_0|f_\infty)\,, \quad t \ge 0}$$

(reduced rate for a defective $\lambda_C^{\min}$: $2(\mu - \varepsilon)$)

Proof:

$$e(t) \overset{\text{CSI}}{\le} \frac{1}{2\lambda_P} S(f(t)) \overset{\text{decay}}{\le} \frac{1}{2\lambda_P} \mathrm{e}^{-2\mu(t-\delta)} S(f(\delta)) \overset{\text{regulariz.}}{\le} c(\delta)\, e^{-2\mu t} \mathrm{e}\,(0)$$

$\square$

Remark: Rate μ is sharp, but constant c is not.

References:

- [Villani] 2009: exponential decay in weighted H^1, but *no sharp rates*
- [Dolbeault-Mouhot-Schmeiser] Trans. AMS 2014: kinetic models, exponential decay in modified L^2–norm
- [Gadat-Miclo] KRM 2013: *sharp rates* for 2 Fokker-Planck toy models
- [Baudoin] 2014: Γ_2–formalism, includes auxiliary gradient functional

sharpness of the exponential decay 1

Theorem 7

Let $\mu := \min\{\Re\,\lambda_C\}$. $\Rightarrow$

1. *If* $\lambda_C^{\min} \in \mathbb{R}$ *with eigenvector* $v_0 \in \mathbb{R}^n$*:*

$$f_0(x) := f_\infty(x)\, e^{\,v_0\cdot x - \frac{v_0^\top \mathbf{K} v_0}{2}} \quad \textit{yields}$$

$$e_1(f(t)|f_\infty) = e^{-2\mu t}\, e_1(f_0|f_\infty)\,. \tag{8}$$

sharpness of the exponential decay 1

Theorem 7

Let $\mu := \min\{\Re\,\lambda_C\}$. $\quad\Rightarrow$

1. *If* $\lambda_C^{\min} \in \mathbb{R}$ *with eigenvector* $v_0 \in \mathbb{R}^n$:

$$f_0(x) := f_\infty(x)\, e^{\,v_0\cdot x - \frac{v_0^\top \mathbf{K} v_0}{2}} \quad \textit{yields}$$

$$e_1(f(t)|f_\infty) = e^{\,-2\mu t}\, e_1(f_0|f_\infty)\,. \tag{8}$$

$$g_0(x) := f_\infty(x)\,(1 + v_0\cdot x) \quad \textit{yields}$$

$$e_2(g(t)|f_\infty) = e^{\,-2\mu t}\, e_2(g_0|f_\infty)\,. \tag{9}$$

sharpness of the exponential decay 1

Theorem 7

Let $\mu := \min\{\Re\,\lambda_C\}$. $\quad\Rightarrow$

1. *If* $\lambda_C^{\min} \in \mathbb{R}$ *with eigenvector* $v_0 \in \mathbb{R}^n$*:*

$$f_0(x) := f_\infty(x)\, e^{\,v_0\cdot x - \frac{v_0^\top \mathbf{K} v_0}{2}} \quad \textit{yields}$$

$$e_1(f(t)|f_\infty) = e^{-2\mu t}\, e_1(f_0|f_\infty)\,. \tag{8}$$

$$g_0(x) := f_\infty(x)\,(1 + v_0\cdot x) \quad \textit{yields}$$

$$e_2(g(t)|f_\infty) = e^{-2\mu t}\, e_2(g_0|f_\infty)\,. \tag{9}$$

2. *If* $\lambda_C^{\min} \notin \mathbb{R}$ *with eigenvector* $v_0 \in \mathbb{C}^n$*:*
$\exists\, f_0,\, g_0$ *such that* "... $\le$ *const.* ..." *in* (8), (9)*;*
and equality for $t = t_0 + n\tau$ *(with some* $t_0 \ge 0,\ \tau > 0,\ n \in \mathbb{N}_0$*).*

sharpness of the exponential decay 2

Remark:

- General entropies: bounded below by $c_1\, e_1$, bounded above by $c_2\, e_2$.

  $\Rightarrow$ Decay rate from Theorem 6 is sharp $\forall$ admissible entropies e_ψ.
- for a defective $\lambda_C^{\min}$: $\exists\ f_0$ with

$$e_{1,2}(t) = \mathrm{e}^{\,-2\mu t}\,(c_0 + c_1\, t + c_2\, t^2)\,.$$

spectrum of L in $L^2(\mathbb{R}^n, f_\infty^{-1})$; L ... non-symmetric

Theorem 8

1. $$\sigma(L) = \sigma_p(L) = \left\{ -\sum_{j=1}^{n} \alpha_j \lambda_j \middle| \alpha \in \mathbb{N}_0^n \right\} \subset \{0\} \cup (\mathbb{R}^- \times i\mathbb{R})$$

with $\{\lambda_j\}$ ... *eigenvalues of* $\mathbf{C}$.

spectrum of L in $L^2(\mathbb{R}^n, f_\infty^{-1})$; L ... non-symmetric

Theorem 8

1.
$$\sigma(L) = \sigma_p(L) = \left\{ -\sum_{j=1}^{n} \alpha_j \lambda_j \middle| \alpha \in \mathbb{N}_0^n \right\} \subset \{0\} \cup (\mathbb{R}^- \times i\mathbb{R})$$

   with $\{\lambda_j\}$ ... *eigenvalues of* $\mathbf{C}$.

2. $L^2(\mathbb{R}^n, f_\infty^{-1})$ *has an orthogonal decomposition in* e^{Lt}*–invariant subspaces, defined by* $\{|\alpha| = const.\}$.
   They are spanned by (generalized) eigenfunctions of L which lie in $\mathcal{P}(\mathbb{R}^n) \cdot f_\infty$.

spectrum of L in $L^2(\mathbb{R}^n, f_\infty^{-1})$; L ... non-symmetric

Theorem 8

1. $$\sigma(L) = \sigma_p(L) = \left\{ -\sum_{j=1}^{n} \alpha_j \lambda_j \middle| \alpha \in \mathbb{N}_0^n \right\} \subset \{0\} \cup (\mathbb{R}^- \times i\mathbb{R})$$

   with $\{\lambda_j\}$ ... eigenvalues of $\mathbf{C}$.

2. *$L^2(\mathbb{R}^n, f_\infty^{-1})$ has an orthogonal decomposition in e^{Lt}–invariant subspaces, defined by $\{|\alpha| = const.\}$.*
   They are spanned by (generalized) eigenfunctions of L which lie in $\mathcal{P}(\mathbb{R}^n) \cdot f_\infty$.

Proof: regularization of e^{Lt} $\Rightarrow$ resolvent is compact in $L^2(\mathbb{R}^n, f_\infty^{-1})$;

$$R(\lambda, L): \ L^2(f_\infty^{-1}) \to \mathcal{H}_r := \{f \mid \frac{f}{f_\infty} \in H^r(\mathbb{R}^n, f_\infty)\}, \ r = \frac{1}{m+1}$$