A GRADIENT FLOW INTERPRETATION OF THE KELLER-SEGEL SYSTEMS

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Entropy Methods, PDEs, Functional Inequalities, and Applications

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GENERAL MODEL

$$\frac{\partial \rho}{\partial t} = \Delta(\rho^m) - \chi \mathrm{div}\left(\rho \nabla \mathcal{K} * \rho\right) \quad \text{in } (0, +\infty) \times \mathbb{R}^d \,, \tag{1}$$

where $\mathcal{K}$ is a given attractive interaction potential and χ is a given constant.

Remark:

$$\int_{\mathbb{R}^d} \rho(x,t)\ \mathrm{d}x = \int_{\mathbb{R}^d} \rho_0(x)\ \mathrm{d}x =: 1$$

In dimension 2, take $\mathcal{K} := -\frac{1}{2\pi}\log|\cdot|$, the Poisson kernel:

THE PATLAK-KELLER-SEGEL SYSTEM

$$\begin{cases} \dfrac{\partial\rho}{\partial t} = \Delta\rho - \chi\mathrm{div}(\rho\nabla\Phi) & \text{in } (0,+\infty)\times\mathbb{R}^2 \\ \Delta\Phi = -\rho & \text{in } (0,+\infty)\times\mathbb{R}^2 . \end{cases} \tag{KS}$$

THE FREE ENERGY FUNCTIONAL

$$\mathcal{F}_{\text{PKS}}[\rho] = \int_{\mathbb{R}^2}\rho(x)\log\rho(x)\,\mathrm{d}x + \frac{\chi}{4\pi}\iint_{\mathbb{R}^2\times\mathbb{R}^2}\rho(x)\log|x-y|\rho(y)\,\mathrm{d}x\,\mathrm{d}y\ .$$

Result: The solutions to the system (KS) can be seen as a gradient flow of the free energy in the Monge-Kantorovich metric:

$$\rho_t = -\nabla_W\mathcal{F}_{\text{PKS}}[\rho(t)]\ .$$

THE JORDAN-KINDERLEHRER-OTTO (JKO) SCHEME

Given a time step h, we define the solution by the minimising scheme:

$$\rho_\tau^{n+1} \in \operatorname{argmin}_{\rho\in\mathcal{K}} \left[\frac{\mathcal{W}_2^2(\rho,\rho_\tau^n)}{2h} + \mathcal{F}_{\mathrm{PKS}}[\rho] \right] ,$$

where $\mathcal{S} := \{\rho \in \mathcal{P}_2(\mathbb{R}^2) \ : \ \int_{\mathbb{R}^2} \rho \log \rho < \infty\}$.

It is classical to prove using the logarithmic Hardy-Littlewood-Sobolev inequality that this minimisation problem has a minimiser if $\chi < 8\pi$.

Let ρ be a minimiser. Let $\zeta \in \mathcal{C}_0^\infty(\mathbb{R}^d;\mathbb{R}^d)$. For $\delta \in (0,1)$, define

$$\rho_\delta \quad := T_\delta \# \rho \quad \text{where} \quad T_\delta := \mathrm{id} + \delta\,\zeta\ .$$

THE DISCRETE EULER-LAGRANGE EQUATION

$$\frac{1}{h}\int_{\mathbb{R}^2} \zeta\,\left[\rho - \rho_h^n\right]\,\mathrm{d}x = \int_{\mathbb{R}^2} \Delta\zeta\,\mathrm{d}\rho - \chi \iint_{\mathbb{R}^2\times\mathbb{R}^2} \frac{[\nabla\zeta(x) - \nabla\zeta(y)]\cdot(x-y)}{|x-y|^2}\,\mathrm{d}\rho(x)\,\mathrm{d}\rho(y) + O(h^{1/2}).$$

The convergence of the scheme

For each positive integer n, let $\nabla\varphi^n$ be the optimal transportation plan with $\nabla\varphi^n \# \rho_h^n = \rho_h^{n-1}$. Then for $(n-1)h \le t \le nh$ we define

McCann's interpolant

$$\rho_h(t) = \left(\frac{t-(n-1)h}{h} \mathrm{id} + \frac{nh-t}{h} \nabla\varphi^n \right) \# \rho_h^n \,.$$

And we have to pass to the limit in the discrete Euler-Lagrange equation:

$$\int_{\mathbb{R}^2} \zeta(x) \left[\rho_h(t_2, x) - \rho_h(t_1, x) \right] \mathrm{d}x = \int_{t_1}^{t_2} \int_{\mathbb{R}^2} \Delta\zeta(x) \, \mathrm{d}\rho_h(s,x) \, \mathrm{d}s$$
$$- \chi \int_{t_1}^{t_2} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{[\nabla\zeta(x) - \nabla\zeta(y)] \cdot (x-y)}{|x-y|^2} \, \mathrm{d}\rho_h(s,y) \, \mathrm{d}\rho_h(s,x) \, \mathrm{d}s + O(h^{1/2}).$$

By the energy and Hölder estimates, up to the extraction of a sub-sequence, $(\rho_h)_h$ converges in $\mathcal{C}([0,T], \mathcal{P}(\mathbb{R}^2))$ and $(\rho_h(t))_h$ in w-$\mathrm{L}^1(\mathbb{R}^2)$. We can pass to the limit and obtain

Weak solutions, [B., Calvez, Carrillo, SINUM 2008]

$$\int_{\mathbb{R}^2} \zeta(x) \left[\rho(t_2, x) - \rho(t_1, x) \right] \mathrm{d}x = \int_{t_1}^{t_2} \int_{\mathbb{R}^2} \Delta\zeta(x) \, \mathrm{d}\rho(s,x) \, \mathrm{d}s$$
$$- \chi \int_{t_1}^{t_2} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{[\nabla\zeta(x) - \nabla\zeta(y)] \cdot (x-y)}{|x-y|^2} \, \mathrm{d}\rho(s,y) \, \mathrm{d}\rho(s,x) \, \mathrm{d}s.$$

THE PARABOLIC-PARABOLIC KELLER-SEGEL SYSTEM

$$\begin{cases} \dfrac{\partial \rho}{\partial t} = \Delta \rho - \chi \mathrm{div}\left[\rho \nabla \phi\right] , \\ \tau \dfrac{\partial \phi}{\partial t} = \Delta \phi - \alpha\, \phi + \rho \,, & (t,x) \in (0,\infty) \times \mathbb{R}^2 \,, \\ 0 \leq \rho_0 \in \mathrm{L}^1(\mathbb{R}^2) \qquad \phi_0 \in \mathrm{H}^1(\mathbb{R}^2) \end{cases} \tag{2}$$

THE FREE ENERGY FUNCTIONAL

$$\mathcal{E}_\alpha[\rho,\phi] := \int_{\mathbb{R}^2} \left\{ \frac{\rho(x)\log\rho(x)}{\chi} - \rho(x)\,\phi(x) + \frac{1}{2}\,|\nabla \phi(x)|^2 + \frac{\alpha}{2}\,\phi(x)^2 \right\} \mathrm{d}x \,.$$

Result: The system (2) has the following “gradient flow” structure

$$\begin{cases} \partial_t \rho = \nabla \cdot \left(\rho \nabla \dfrac{\delta \mathcal{E}_\alpha}{\delta \rho} \right) \\ \partial_t \phi = -\dfrac{\delta \mathcal{E}_\alpha}{\delta \phi} \end{cases} ,$$

Define

$$\mathcal{K} := \{\rho \in \mathcal{P}_2(\mathbb{R}^2) \ : \ \int_{\mathbb{R}^2} \rho \log \rho < \infty\} \, .$$

THE HYBRID VARIATIONAL SCHEME

$$\begin{cases} (\rho_h^0, \phi_h^0) = (\rho_0, \phi_0) \, , \\ (\rho_h^{n+1}, \phi_h^{n+1}) \in \mathrm{Argmin}_{(\rho,\phi)\in\mathcal{K}} \mathcal{F}_{h,n}[\rho,\phi] \, , \qquad n \geq 0 \, , \end{cases} \tag{3}$$

where

$$\mathcal{F}_{h,n}[\rho,\phi] := \frac{1}{2h} \left[\frac{\mathcal{W}_2^2(\rho,\rho_h^n)}{\chi} + \tau \, \|\phi - \phi_h^n\|_2^2 \right] + \mathcal{E}_\alpha[\rho,\phi] \, .$$

To obtain the Euler-Lagrange equation, define

$$\rho_\delta = (\mathrm{id} + \delta\,\zeta)\#\rho\ , \quad \phi_\delta := \phi + \delta\, w\ ,$$

where $\zeta \in \mathcal{C}_0^\infty(\mathbb{R}^2;\mathbb{R}^2)$ and $w \in \mathcal{C}_0^\infty(\mathbb{R}^2)$.

Identifying the Euler-Lagrange equation requires to pass to the limit as $\delta \to 0$ in

$$\frac{\mathcal{W}_2^2(\rho_\delta,\rho_0) - \mathcal{W}_2^2(\rho,\rho_0)}{2\delta} \quad \text{and} \quad \frac{\rho_\delta \log \rho_\delta - \rho \log \rho}{\delta}\ ,$$

which can be performed by standard arguments, but also in

$$\frac{1}{\delta}\int_{\mathbb{R}^2} (\rho\,\phi - \rho_\delta\,\phi_\delta)(x)\,\mathrm{d}x = \int_{\mathbb{R}^2} \rho(x) \left[\frac{\phi(x) - \phi(x+\delta\zeta(x))}{\delta} - w(x+\delta\zeta(x))\right]\,\mathrm{d}x\ .$$

This is where the main difficulty lies: indeed, since $\phi \in \mathrm{H}^1(\mathbb{R}^2)$, we only have

$$\frac{\phi\circ(\mathrm{id} + \delta\zeta) - \phi}{\delta} \rightharpoonup \zeta\cdot\nabla\phi \quad \text{in } \mathrm{L}^2(\mathbb{R}^2),$$

while ρ is only in $\mathcal{K}$.

$$\text{Find } u_n \text{ which minimises} \quad u \mapsto \frac{1}{2h}\mathcal{W}_2^2(u, u_{n-1}) + \mathcal{F}[u] \,. \tag{4}$$

Imagine now that we can find a displacement convex functional $\mathcal{V}$ such that:

$$D^{\mathcal{V}}\mathcal{F}[u] := \limsup_{t\to 0} \frac{\mathcal{F}[u] - \mathcal{F}[S_t^{\mathcal{V}} u]}{t} \,.$$

is well-defined where $(S_t^{\mathcal{V}})_{t\geq 0}$ is the Monge-Kantorovich gradient flow associated to $\mathcal{V}$. Taking $u = S_t^{\mathcal{V}} u_n$ in (4) we obtain

$$\mathcal{F}[u_n] - \mathcal{F}[S_t^{\mathcal{V}} u_n] \leq \frac{1}{2h}\left[\mathcal{W}_2^2(S_t^{\mathcal{V}} u_n, u_{n-1}) - \mathcal{W}_2^2(u_n, u_{n-1})\right] \,.$$

Dividing by t and letting $t \to 0$ and as $\mathcal{V}$ is displacement convex, we obtain

MMS' TYPE DISCRETE ESTIMATE

$$D^{\mathcal{V}}\mathcal{F}[u_n] \leq \frac{1}{2h}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{W}_2^2(S_t^{\mathcal{V}} u_n, u_{n-1}) \leq \frac{\mathcal{V}[u_{n-1}] - \mathcal{V}[S_t^{\mathcal{V}} u_n]}{h} \,.$$

FRAMEWORK

Consider the classical parabolic-elliptic Patlak-Keller-Segel system when $\chi = 8\pi$ and the 2-moment is unbounded.

THE "CRITICAL" NONLINEAR FOKKER-PLANCK EQUATION

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta\left(\sqrt{u}\right) + \dfrac{1}{\sqrt{2\lambda}}\,\mathrm{div}(x\,u) & t > 0\,,\ x \in \mathbb{R}^2\,, \\ u(0) = u_0 \geq 0 & x \in \mathbb{R}^2\,, \end{cases} \tag{5}$$

Define

FAST DIFFUSION FUNCTIONAL

$$\mathcal{H}_\lambda[u] := \int_{\mathbb{R}^2} \frac{\left(\sqrt{u} - \sqrt{\bar\rho_\lambda}\right)^2}{\sqrt{\bar\rho_\lambda}}\,\mathrm{d}x$$

It follows that for classical solutions u of (5) ,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_\lambda[u(t)] = -\int_{\mathbb{R}^2} u(t,x)\left|\nabla\left(\frac{1}{\sqrt{\bar\rho_\lambda}} - \frac{1}{\sqrt{u}}\right)\right|^2 \mathrm{d}x \leq 0\,.$$

The $\bar\rho_\lambda$ are stationary solutions of (KS).

If ρ is the smooth solution to (KS) with $\chi = 8\pi$ we obtain

$$\mathcal{D}[\rho(t)] := \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_\lambda[\rho(t)] = -8\int_{\mathbb{R}^2} |\nabla(\rho^{1/4})|^2\,\mathrm{d}x + \int_{\mathbb{R}^2} \rho^{3/2}\,\mathrm{d}x\ .$$

GAGLIARDO-NIRENBERG-SOBOLEV INEQUALITY, [DEL PINO, DOLBEAULT]

For all functions f in $\mathbb{R}^2$ with a square integrable distributional gradient ∇f,

$$\pi\int_{\mathbb{R}^2} |f|^6\,\mathrm{d}x \le \int_{\mathbb{R}^2} |\nabla f|^2\,\mathrm{d}x \int_{\mathbb{R}^2} |f|^4\,\mathrm{d}x\ ,$$

and there is equality if and only if f is a multiple of a translate of $\bar{\rho}_\lambda^{1/4}$ for some $\lambda > 0$.

DISSIPATION OF $\mathcal{H}_\lambda$

For all solution ρ to (KS) of mass $\chi = 8\pi$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_\lambda[\rho] = \mathcal{D}[\rho(t)] \le 0\ ,$$

and moreover, there is equality if and only ρ is a translate of $\bar{\rho}_\lambda$ for some $\lambda > 0$.

Using the MMS' technique with $\mathcal{F}_{\mathrm{PKS}}$ and the displacement convex functional $\mathcal{H}_\lambda$ gives

Above the tangent formulation

$$\frac{1}{2}\int_{\mathbb{R}^2} \frac{\nabla u_h^n}{(u_h^n)^{3/2}}\,\mathrm{d}x - \int_{\mathbb{R}^2} (u_h^n)^{3/2}\,\mathrm{d}x \leq \frac{\mathcal{H}_\lambda[u_h^n] - \mathcal{H}_\lambda[u_h^{n+1}]}{h}\,.$$

GLOBAL EXISTENCE AND LARGE TIME BEHAVIOUR

Given any density ρ_0 with total mass 8π such that there exists $\lambda > 0$ with

$$\mathcal{H}_\lambda[\rho_0] < \infty.$$

Then there exists $\rho \in \mathcal{AC}^0([0,T], \mathcal{P}_2(\mathbb{R}^2))$, with $\rho(t) \in L^1(\mathbb{R}^2)$ for all $t \geq 0$ being a global-in-time weak solution of (KS). Moreover, the solutions constructed satisfy

$$\mathcal{F}_{\mathrm{PKS}}[\rho(t)] \leq \mathcal{F}_{\mathrm{PKS}}[\rho_0]\,,$$

and

$$\mathcal{H}_\lambda[\rho(t)] + \int_0^t \mathcal{D}[\rho(t)]\,\mathrm{d}t \leq \mathcal{H}_\lambda[\rho_0]\,.$$

Furthermore,

$$\lim_{t\to\infty} \mathcal{F}_{\mathrm{PKS}}[\rho(t)] = \mathcal{F}_{\mathrm{PKS}}[\bar\rho_\lambda] \qquad \lim_{t\to\infty} \|\rho(t) - \bar\rho_\lambda\|_{L^1(\mathbb{R}^2)} = 0\,.$$

And the system satisfies the hypercontractivity property *i.e.* for any $t^* > 0$, the constructed solution ρ is bounded in $L^\infty(t^*, \infty, L^p(\mathbb{R}^2))$, for any $p \in (1,\infty)$.

Talagrand's inequality:

$$\mathcal{W}_2^2(\rho, \bar\rho_\lambda) \leq 2\sqrt{2\lambda}\,\mathcal{H}_\lambda[\rho]\,.$$

Basin on attraction: If $\lambda \neq \mu$ then

$$\mathcal{W}_2(\bar\rho_\mu, \bar\rho_\lambda) = \frac{1}{2}\int_{\mathbb{R}^2} \left|\frac{\lambda}{\mu}x - x\right|^2 \bar\rho_\mu = +\infty\,.$$

THE PARABOLIC-PARABOLIC KELLER-SEGEL SYSTEM IN HIGHER DIMENSIONS

$$\begin{cases} \partial_t \rho = \Delta \rho^m - \chi \mathrm{div}\,[\rho \nabla \phi]\,, \\ \tau \partial_t \phi = \Delta \phi - \alpha\,\phi + \rho\,, \end{cases} \quad (t,x) \in (0,\infty) \times \mathbb{R}^d\,, \tag{6}$$

THE FREE ENERGY

$$\mathcal{F}[\rho] := \int_{\mathbb{R}^d} \frac{\rho^m(t,x)}{m-1}\,\mathrm{d}x - \chi \frac{c_d}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{\rho(t,x)\,\rho(t,y)}{|x-y|^{d-2}}\,\mathrm{d}x\,\mathrm{d}y\,.$$

MAIN DIFFICULTY

Above the choice of the auxiliary gradient flow naturally comes from the existence of another Liapunov functional which is different from the energy. Such a nice structure does not seem to be available for our problem.

We try to apply MMS' technique with, where we take here $\alpha = 0$ to simplify the exposition,

$$\mathcal{H}[f,g] := \int_{\mathbb{R}^d} \left(f(x) \log f(x) + \frac{1}{2} \int_{\mathbb{R}^d} |\nabla g(x)|^2 \right)$$

and f, g the associated flow. We compute

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{E}_0[f,g] = \underbrace{-\frac{4}{m} \|\nabla \left(f^{m/2}\right)\|_2^2 - \|\Delta g + f\|_2^2}_{:=\mathcal{D}[f,g]} + \|f\|_2^2\,, \quad t > 0\,.$$

Using the Hölder and Sobolev inequalities:

$$\|w\|_2^2 \leq \|w\|_m \, \|w\|_{m/(m-1)} \leq C \, \|w\|_m \, \left\| \nabla(|w|^{m/2}) \right\|_2^{2/m} .$$

Combining the above estimate with Young's inequality gives

$$\|\rho_h^n\|_2^2 \leq \frac{2}{m} \left\| \nabla[(\rho_h^n)^{m/2}] \right\|_2^2 + C \, \|\rho_h^n\|_m^{m/(m-1)} ,$$

and thus

$$\|\rho_h^n\|_2^2 \leq \frac{1}{2} \mathcal{D}[\rho_h^n, \phi_h^n] + C \, \|\rho_h^n\|_m^{m/(m-1)} .$$

By the MMS estimate we obtain, for any $\chi < \chi_c$

$$\|\rho_h^n\|_2^2 \leq \frac{1}{2} \mathcal{D}[\rho_h^n, \phi_h^n] + C \, \mathcal{E}_0[\rho_h^n, \phi_h^n]^{1/(m-1)} .$$

And finally (7) implies

DISCRETE ESTIMATE

$$\frac{1}{2} \mathcal{D}[\rho_h^n, \phi_h^n] \leq \frac{\mathcal{H}[\rho_h^{n-1}, \phi_h^{n-1}] - \mathcal{H}[\rho_h^n, \phi_h^n]}{h} + C \, \mathcal{E}_0[\rho_h^n, \phi_h^n]^{1/(m-1)} .$$

Consider the free energy

$$\mathcal{G}[u,v] := \int_{\mathbb{R}^2} \left\{ u(x)\log u(x) + \frac{\alpha}{2} v(x)^2 + \frac{1}{2}|\nabla v(x)|^2 \right\} \mathrm{d}x \ .$$

The flow associated to this energy is given by the solution (u, v) to

$$\begin{cases} \partial_s u &= \Delta u \quad \text{in } (0,\infty)\times\mathbb{R}^2, \quad u(0)=\rho \\ \partial_s v &= \Delta v - \alpha v \quad \text{in } (0,\infty)\times\mathbb{R}^2, \quad v(0)=\phi \end{cases} \tag{8}$$

We compute

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}_\alpha[u,v] = \underbrace{-\frac{4}{\chi}\|\nabla\sqrt{u}\|_2^2 - \|\Delta v + u - \alpha v\|_2^2}_{=:-\mathcal{D}[u,v]} + \underbrace{\|u\|_2^2 - \alpha\int_{\mathbb{R}^2} uv\,\mathrm{d}x}_{=\Re[u,v]}$$

We are not in the situation of MMS' technique but we can control $\Re[u,v]$.

$$2\int_{\mathbb{R}^2} u(\sigma)\, v(\sigma)\,\mathrm{d}x \le \|u(\sigma)\|_2^2 + \|v(\sigma)\|_2^2$$

we have

$$\Re[u(\sigma), v(\sigma)] \le \left(1+\frac{\alpha}{2}\right)\|u(\sigma)\|_2^2 + \frac{\alpha}{2}\|v(\sigma)\|_2^2 \,.$$

We focus on the control of $\|u(\sigma)\|_2^2$. Applying Biler-Hebisch-Nadzieja's estimate to $\sqrt{u(\sigma)}$ gives

$$\begin{aligned}\|u(\sigma)\|_2^2 = \|\sqrt{u(\sigma)}\|_4^4 &\le \frac{\varepsilon}{2}\|\nabla\sqrt{u(\sigma)}\|_2^2\|u(\sigma)\log u(\sigma)\|_1 + L_{\varepsilon/2}\|u(\sigma)\|_1 \\ &\le \frac{\varepsilon}{2}\|u(\sigma)\log u(\sigma)\|_1\,\frac{\chi}{4}\,\mathcal{D}[u(\sigma), v(\sigma)] + C_\varepsilon\end{aligned} \tag{9}$$

Using the properties of the heat equation we end up with

$$\|u(\sigma)\|_2^2 \le \frac{\varepsilon\chi}{8}\,\Lambda\,\mathcal{D}[u(\sigma), v(\sigma)] + C_\varepsilon \,, \quad \sigma\in(0,1]\,.$$

for

$$\Lambda \ge \int_{\mathbb{R}^2}\rho\log\rho\,\mathrm{d}x + \frac{2}{\mathrm{e}} + 16\,.$$

We conclude

$$\frac{4}{\chi}\|\nabla\sqrt{\rho}\|_2^2 + \|\Delta\phi+\rho-\alpha\phi\|_2^2 \le \frac{\mathcal{G}[\rho,\phi]-\mathcal{G}[\rho_0,\phi_0]}{h} + C_4(\Lambda) + \alpha\|\phi\|_2^2\,.$$

Merci pour votre attention