Long time asymptotics of a porous medium equation with fractional pressure

Yanghong Huang

Department of Mathematics
Imperial College London

joint work with:
J.A. Carrillo (Imperial), M. C. Santos (U. Estadual de Campinas), J.L. Vazquez (Madrid)

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The problem formulation

- The porous medium equation with fractional pressure by L. Caffarelli and J. L. Vázquez (2011):

$$u_\tau = \nabla \cdot (u \nabla p), \quad p = (-\Delta)^{-s} u,$$

on $(\tau, y) \in \mathbb{R}^+ \times \mathbb{R}^d$.

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- Long time asymptotic behaviours: equation in similarity variables $x = y(1+\tau)^{-\beta}, t = \log(1+\tau), \rho = (1+\tau)^\alpha u$:

$$\rho_t = \nabla \cdot \left(\rho(\nabla(-\Delta)^{-s}\rho + \beta x)\right),$$

with $\alpha = d/(d+2-2s), \alpha = 1/(d+2-2s)$.

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  with $\alpha = d/(d+2-2s), \alpha = 1/(d+2-2s)$.

- The explicit steady state $\rho_\infty(x) = K_{d,s}(R^2 - |x|^2)_+^{1-s}$ by Biler, Imbert and Karch (2011).

Definitions and notations for $\rho_t = \nabla \cdot \big(\rho(\nabla(-\Delta)^{-s}\rho + \lambda x)\big)$

- The Riesz operator $(-\Delta)^{-s}$ wit $s \in (0,1)$

$$(-\Delta)^{-s}\rho(x) = c_{d,s}\int_{\mathbb{R}^d} |x-y|^{2s-d}\rho(y)\,dy, \quad c_{d,s} = \frac{s2^{-2s}\Gamma(d/2-s)}{\pi^{d/2}\Gamma(1+s)}.$$

and the fractional Laplacian $(-\Delta)^s$

$$(-\Delta)^s\rho(x) = -c_{d,-s}\int_{\mathbb{R}^d} \frac{\rho(x)-\rho(y)}{|x-y|^{d+2s}}\,dy.$$

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- The energy (entropy) and its dissipation:

$$\mathcal{E}[\rho] = \frac{1}{2}\int \rho(-\Delta)^{-s}\rho + \frac{\lambda}{2}\int |x|^2\rho,$$
$$\mathcal{I}[\rho] = \int \rho|\nabla\xi|^2, \quad \xi = \frac{\delta\mathcal{E}[\rho]}{\delta\rho} = (-\Delta)^{-s}\rho + \frac{\lambda}{2}|x|^2.$$

Outline of the talk

$$\rho_t = \nabla \cdot \left(\rho(\nabla(-\Delta)^{-s}\rho + \lambda x)\right)$$

Goal: examine the convergence of $\rho(t)$ to ρ_∞ through the decay of $\mathcal{E}[\rho(t)] - \mathcal{E}[\rho_\infty]$

- Key inequality: $\mathcal{E}[\rho] - \mathcal{E}[\rho_\infty] \leq \mathcal{I}[\rho]/2\lambda$ (combined with $\frac{d}{dt}\mathcal{E}[\rho(t)] = -\mathcal{I}[\rho(t)]$)
- The three approaches (via formal computation)
  a) Entropy dissipation method
  b) Fractional Sobolev inequalities
  c) Transport inequalities
- Regularization and exponential convergence

Entropy dissipation method [1]

Compare the first and the second order derivative of $\mathcal{E}[\rho]$:

- The first order derivative is $\frac{d}{dt}\mathcal{E}[\rho(t)] = -\mathcal{I}[\rho(t)]$
- Write the second order derivative as

$$\frac{d}{dt}\mathcal{I}[\rho(t)] = -2\lambda\mathcal{I}[\rho(t)] - 2\mathcal{R}[\rho(t)].$$

[1] A. Arnold, J. A. Carrillo ,A. Jüngel, P. Markowich, R. J. McCann ,G. Toscani, and A. Unterreiter.,C. Villani,$\cdots$

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If $\mathcal{R}[\rho(t)]$ is **non-negative**, then $\dfrac{d}{dt}\mathcal{I}[\rho(t)] \leq -2\lambda\mathcal{I}[\rho(t)]$
$\Longrightarrow$ exponential convergence of $\mathcal{E}[\rho](t) - \mathcal{E}[\rho_\infty]$ and $\mathcal{I}[\rho](t)$:

$$\mathcal{E}[\rho](t) - \mathcal{E}[\rho_\infty] \leq \left(\mathcal{E}[\rho](0) - \mathcal{E}[\rho_\infty]\right) e^{-2\lambda t},$$
$$\mathcal{I}[\rho(t) \leq \mathcal{I}[\rho(0)] e^{-2\lambda t}.$$

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Entropy dissipation method: Two examples

- The porous medium equation with quadratic power ($s = 0$): $\rho_t = \nabla \cdot (\rho(\nabla \rho + \lambda x))$,

$$\mathcal{E}[\rho] = \frac{1}{2}\int \rho^2 + \frac{\lambda}{2}\int |x|^2\rho, \quad \mathcal{I}[\rho] = \int \rho|\nabla\xi|^2,\ \xi = \rho + \frac{\lambda}{2}|x|^2.$$

$$\Longrightarrow \mathcal{R}[\rho] = \tfrac{1}{2}\int \rho((\Delta\xi)^2 + \|D^2\xi\|_F^2) \geq 0.$$

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- Equation from granular flow or aggregation: $\rho = \nabla \cdot (\rho \nabla W * \rho)$ with W relatively smooth (and $\lambda = 0$):

$$\mathcal{E}[\rho] = \frac{1}{2}\int \rho W * \rho, \quad \mathcal{I}[\rho] = \int \rho|\nabla\xi|^2, \quad \xi = W * \rho.$$

$$\mathcal{R}[\rho] = \int\int \rho(x)\rho(y)\langle D^2 W(x-y)(\nabla\xi(x) - \nabla\xi(y)), \nabla\xi(x) - \nabla\xi(y)\rangle$$

Entropy dissipation method: the fractional PME

Write the equation as $\rho_t = \nabla \cdot (\rho \nabla \xi)$ with $\xi = \frac{\delta \mathcal{E}[\rho]}{\delta \rho} = (-\Delta)^{-s}\rho + \frac{\lambda}{2}|x|^2$, then $D^2\xi = D^2(-\Delta)^{-s}\rho + \lambda I, \quad \xi_t = (-\Delta)^{-s}\rho_t = (-\Delta)^{-s}[\nabla \cdot (\rho \nabla \xi)]$.

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Step 1: Take the derivative $\frac{d}{dt}\mathcal{I}[\rho] = \frac{d}{dt}\int \rho|\nabla\xi|^2$

$$\int \rho_t|\nabla\xi|^2 + 2\int \rho\nabla\xi \cdot \nabla\xi_t$$

$$= \int \nabla \cdot (\rho\nabla\xi)|\nabla\xi|^2 + 2\int \rho\nabla\xi \cdot \nabla\Big[(-\Delta)^{-s}\big(\nabla \cdot (\rho\nabla\xi)\big)\Big]$$

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Step 2: Extract $-2\lambda \mathcal{I}[\rho]$ from the first integral

$$\int \nabla \cdot (\rho \nabla \xi)|\nabla \xi|^2 = -2\int \rho \langle D^2\xi \cdot \nabla \xi, \nabla \xi \rangle$$

$$= -2\lambda \mathcal{I}(\rho) - 2\int \rho \big\langle D^2(-\Delta)^{-s}\rho \cdot \nabla \xi, \nabla \xi \big\rangle$$

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Step 3: Obtain the expression for $\mathcal{R}[\rho]$ as

$$\int \rho\Big\langle D^2(-\Delta)^{-s}\rho\cdot\nabla\xi,\nabla\xi\Big\rangle - \int \rho\nabla\xi\cdot\nabla(-\Delta)^{-s}\Big[\nabla\cdot(\rho\nabla\xi)\Big]$$

Entropy dissipation method

$$\mathcal{R}[\rho] = \int \rho \langle D^2 (-\Delta)^{-s} \rho \cdot \nabla \xi, \nabla \xi \rangle - \int \rho \nabla \xi \cdot \nabla (-\Delta)^{-s} [\nabla \cdot (\rho \nabla \xi)]$$

Entropy dissipation method

$$\mathcal{R}[\rho] = \int \rho \langle D^2(-\Delta)^{-s}\rho \cdot \nabla\xi, \nabla\xi \rangle - \int \rho\nabla\xi \cdot \nabla(-\Delta)^{-s}[\nabla \cdot (\rho\nabla\xi)]$$

The basic tools to simplify $\mathcal{R}[\rho]$: the explicit expressions

$$D^2(-\Delta)^{-s}\rho(x) = c_{d,s} \int K_{ij}(x-y)\big(\rho(x)-\rho(y)\big)\, dy$$

where $K_{ij}(x) = (d-2s)|x|^{2s-2-d}\big((d+2-2s)\frac{x_i x_j}{|x|^2} - \delta_{ij}\big) = \partial_{ij}\big(|x|^{2s-d}\big)$.

Entropy dissipation method

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The simplified $\mathcal{R}(\rho)$ is given by

$$\frac{c_{d,s}}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\rho(x)\rho(y)\Big\langle \mathbf{K}(x-y)\Big(\nabla\xi(x)-\nabla\xi(y)\Big), \nabla\xi(x)-\nabla\xi(y)\Big\rangle dydx$$

The eigenvalues of $\mathbf{K}(x-y)$

$$K_{ij}(x) = (d-2s)|x|^{2s-2-d}\Big((d+2-2s)\frac{x_i x_j}{|x|^2} - \delta_{ij}\Big)$$

or

$$\mathbf{K} = (d-2s)|x|^{2s-2-d}\left((d+2-2s)\frac{x\otimes x}{|x|^2} - I\right)$$

The eigenvalues are (without the common factor $d-2s$)

$$(d+1-2s)|x|^{2s-2-d},\ -|x|^{2s-2-d},\ \cdots,\ -|x|^{2s-2-d}.$$

$\frac{c_{d,s}}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\rho(x)\rho(y)\Big\langle \mathbf{K}(x-y)\Big(\nabla\xi(x)-\nabla\xi(y)\Big), \nabla\xi(x)-\nabla\xi(y)\Big\rangle dydx$ is nonnegative only in one dimension.

$\Longrightarrow$ Exponential convergence in one dimension.

Generalized Sobolev Inequalities

- Basic idea: Prove the inequality $\mathcal{E}[\rho] - \mathcal{E}[\rho_\infty] \leq \frac{1}{2\lambda}\mathcal{I}[\rho]$
- The quadratic terms $\int |x|^2 \rho$ are canceled
- $\mathcal{E}[\rho_\infty]$ is expressed as a power of $\int \rho_\infty = \int \rho$
- It is easier to prove the resulting inequality in product form
- Reduce to the logarithmic Sobolev inequality for the linear Fokker-Planck equation or Gagliardo-Nirenberg inequalities for the porous medium equation [2]

[2] M. Del Pino and J. Dolbeault. J. Math. Pures Appl. 2002.

Generalized Sobolev Inequalities

$$\frac{1}{2}\int \rho(-\Delta)^{-s}\rho + \frac{\lambda}{2}\int |x|^2\rho - \mathcal{E}[\rho_\infty] \le \frac{1}{2\lambda}\int \rho|\nabla\xi|^2$$
$$= \frac{1}{2\lambda}\int \rho|\nabla(-\Delta)^{-s}\rho|^2 + \int \rho(x)x\cdot\nabla(-\Delta)^{-s}\rho + \frac{\lambda}{2}\int |x|^2\rho$$

- The quadratic terms $\int |x|^2\rho$ on both sides are canceled
- Using the explicit solution [3] $\rho_\infty(x) = K_{d,s}(R^2 - x^2)_+^{1-s}$

$$\mathcal{E}[\rho_\infty] = \tilde{K}_{d,s}\left(\int \rho_\infty(x)\,dx\right)^{\frac{d+4-2s}{d+2-2s}} = \tilde{K}_{d,s}\left(\int \rho(x)\,dx\right)^{\frac{d+4-2s}{d+2-2s}}$$

- The cross term on the right hand side (using symmetrization)

$$\int_{\mathbb{R}^d} \rho(x)x\cdot\nabla(-\Delta)^{-s}\rho(x)\,dx = \frac{2s-d}{2}\int_{\mathbb{R}^d} \rho(-\Delta)^{-s}\rho$$

[3] P. Biler, C. Imbert, and G. Karch. C. R. Math. Acad. Sci. 2011.

Generalized Sobolev Inequalities

We have to prove an inequality relating $\int \rho(-\Delta)^{-s}\rho$, $\int \rho$ and $\int \rho|\nabla(-\Delta)^{-s}\rho|^2$:

$$\lambda(d+1-2s)\int \rho(-\Delta)^{-s}\rho \leq 2\lambda\tilde{K}_{d,s}\left(\int \rho(x)\right)^{\frac{d+4-2s}{d+2-2s}} + \int \rho|\nabla(-\Delta)^{-s}\rho|^2$$

which is equivalent to the inequality in product form

$$\int \rho(-\Delta)^{-s}\rho \leq C_{d,s}\left(\int \rho(x)\,dx\right)^{2-3\theta}\left(\int \rho|\nabla(-\Delta)^{-s}\rho|^2\right)^{\theta}$$

with $\theta = (d-2s)/(2d+4s-2)$.

Transport inequalities [4]

General transport inequalities for a given energy $\mathcal{E}[\rho]$ (and $\mathcal{I}[\rho]$):

- **Log-Sobolev Inequality**

$$\mathcal{E}[\rho] - \mathcal{E}[\rho_\infty] \leq \frac{1}{2\lambda}\mathcal{I}[\rho]$$

- **Talagrand Inequality**

$$W_2(\rho, \rho_\infty) \leq \sqrt{\frac{2}{\lambda}\left(\mathcal{E}[\rho] - \mathcal{E}[\rho_\infty]\right)}$$

- **HWI Inequality**

$$\mathcal{E}(\rho) - \mathcal{E}(\rho_\infty) \leqslant \sqrt{\mathcal{I}(\rho)}\, W(\rho, \rho_\infty) - \frac{\lambda}{2} W^2(\rho, \rho_\infty)$$

Remark: ρ_∞ has to be the minimizer of $\mathcal{E}[\rho]$ only in the first inequality.

[4] F. Otto and C. Villani. *J. Funct. Anal.* 2000.

From HWI to other transport inequalities

From **HWI Inequality** to **Log-Sobolev Inequality**:

$$\begin{aligned}\mathcal{E}(\rho) - \mathcal{E}(\rho_\infty) &\le \sqrt{\mathcal{I}(\rho)}\, W(\rho, \rho_\infty) - \frac{\lambda}{2} W^2(\rho, \rho_\infty) \\ &\le -\frac{\lambda}{2}\left[W(\rho, \rho_\infty) - \frac{1}{\lambda}\sqrt{\mathcal{I}[\rho]}\right]^2 + \frac{1}{2\lambda} I[\rho]\end{aligned}$$

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From **HWI Inequality** to **Talagrand Inequality**: exchange ρ and ρ_∞ and use the fact $\mathcal{I}(\rho_\infty) = 0$,

$$\mathcal{E}(\rho_\infty) - \mathcal{E}(\rho) \le \sqrt{\mathcal{I}(\rho_\infty)}\, W(\rho, \rho_\infty) - \frac{\lambda}{2} W^2(\rho, \rho_\infty) = -\frac{\lambda}{2} W^2(\rho, \rho_\infty).$$

Transport inequalities

The HWI inequality for convex functions:

$$f(w) - f(0) - |w|\ |\nabla f(w)| + \frac{\lambda}{2}|w|^2 \leq f(w) - f(0) - w \cdot \nabla f(w) + \frac{\lambda}{2}|w|^2 \leq 0$$

if $f(w) - \frac{\lambda}{2}|w|^2$ is convex.

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Proof.

(a) The case with $\lambda = 0$:

$$-\Big[f(0) - f(w) - w \cdot \nabla f(w)\Big] \le 0.$$

(b) Replace $f(w)$ by $f(w) - \frac{\lambda}{2}|w|^2$:

$$f(w) - \frac{\lambda}{2}|w|^2 - f(0) - w \cdot \nabla\Big[f(w) - \frac{\lambda}{2}|w|^2\Big] \le 0.$$

□

Transport inequalities

$$\mathcal{E}(\rho) - \mathcal{E}(\rho_\infty) \leqslant \sqrt{\mathcal{I}(\rho)}\, W(\rho, \rho_\infty) - \frac{\lambda}{2} W^2(\rho, \rho_\infty)$$

Essential ingredients of the proof:

- Cauchy-Schwartz inequality to $\sqrt{\mathcal{I}(\rho)}\, W(\rho, \rho_\infty)$
- Inequality for the displacement-convex energy $\mathcal{E}(\rho) - \frac{\lambda}{2} W^2(\rho, \rho_\infty)$
- A few identities in terms of the transport map from ρ_∞ to ρ ($\rho_\infty = T\#\rho$):

$$W_2(\rho, \rho_\infty)^2 = \int |x - T(x)|^2 \rho(x)\, dx, \ \int |x|^2 \rho_\infty(x)\, dx = \int |T(x)|^2 \rho(x)\, dx,$$

The proof of the transport inequalities

$$\mathcal{E}[\rho] - \mathcal{E}[\rho_\infty] \leqslant \sqrt{\mathcal{I}(\rho)}\, W(\rho, \rho_\infty) - \frac{\lambda}{2} W^2(\rho, \rho_\infty)$$

- Cauchy-Schwartz inequality to $\sqrt{\mathcal{I}(\rho)}\, W(\rho, \rho_\infty)$

$$\sqrt{\mathcal{I}[\rho]}\, W(\rho, \rho_\infty) = \sqrt{\int \rho(x) \left|\nabla \frac{\delta \mathcal{E}_1[\rho]}{\delta \rho} + \lambda x\right|^2} \sqrt{\int |x - T(x)|^2 \rho(x)\, dx}$$

$$\geq \int \rho(x) \left(\nabla \frac{\delta \mathcal{E}_1[\rho]}{\delta \rho} + \lambda x\right) \cdot (x - T(x))\, dx$$

where $\mathcal{E}_1[\rho] = \mathcal{E}[\rho] - \frac{\lambda}{2} \int |x|^2 \rho(x)\, dx = \frac{1}{2} \int \rho(-\Delta)^{-s} \rho.$

The proof of the transport inequalities

$$\mathcal{E}[\rho] - \mathcal{E}[\rho_\infty] \leqslant \sqrt{\mathcal{I}(\rho)}\, W(\rho,\rho_\infty) - \frac{\lambda}{2} W^2(\rho,\rho_\infty)$$

- Cauchy-Schwartz inequality to $\sqrt{\mathcal{I}(\rho)}\, W(\rho,\rho_\infty)$

$$\sqrt{\mathcal{I}[\rho]}\, W(\rho,\rho_\infty) = \sqrt{\int \rho(x) \left| \nabla \frac{\delta \mathcal{E}_1[\rho]}{\delta\rho} + \lambda x \right|^2} \sqrt{\int |x - T(x)|^2 \rho(x)\, dx}$$

$$\geq \int \rho(x) \left(\nabla \frac{\delta \mathcal{E}_1[\rho]}{\delta\rho} + \lambda x \right) \cdot (x - T(x))\, dx$$

where $\mathcal{E}_1[\rho] = \mathcal{E}[\rho] - \frac{\lambda}{2} \int |x|^2 \rho(x)\, dx = \frac{1}{2} \int \rho(-\Delta)^{-s} \rho.$

- The HWI inequality is proved if

$$\mathcal{E}_1[\rho] - \mathcal{E}_1[\rho_\infty] \leq \int \rho(x) \nabla \frac{\delta \mathcal{E}_1[\rho]}{\delta\rho} \cdot (x - T(x)),$$

or $\mathcal{E}_1[\rho]$ is displacement-convex.

Displacement convexity of $\mathcal{E}_1[\rho]$

Theorem (R. J. McCann[5]) if W is convex, then the interaction energy $\mathcal{E}_1[\rho] = \frac{1}{2}\iint W(x-y)\rho(x)\rho(y)\,dydx$ is displacement-convex.

[5] R. J. McCann. *Adv. Math.* 1997.

[6] J. A. Carrillo, L. C. F. Ferreira, and J. C. Precioso. *Adv. Math.*, 2012

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But the interaction kernel for

$$\mathcal{E}_1[\rho] = \frac{1}{2}\int \rho(-\Delta)^{-s}\rho = \frac{c_{d,s}}{2}\iint_{\mathbb{R}^d\times\mathbb{R}^d} \rho(x)|x-y|^{2s-d}\rho(y)$$

is not convex in general.

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But it is possible in one dimension [6]:

a) the monotonicity of the transport map and

b) the convexity of the kernel on the positive axis.

[5] R. J. McCann. *Adv. Math.* 1997.

[6] J. A. Carrillo, L. C. F. Ferreira, and J. C. Precioso. *Adv. Math.*, 2012

The final inequality with $\mathcal{E}_1[\rho] = \int \rho(-\Delta)^{-s}\rho$

$$\mathcal{E}_1[\rho] - \mathcal{E}_1[\rho_\infty] \leq \int \rho(x)\nabla\frac{\delta\mathcal{E}_1[\rho]}{\delta\rho}\cdot(x - T(x))$$

With the transport map ($k_s(x) = |x|^{2s-1}$)

$$\mathcal{E}_1[\rho_\infty] = \frac{c_{1,s}}{2}\int \rho_\infty(x)k_s(x-y)\rho_\infty(y) = \frac{c_{1,s}}{2}\int \rho(x)k_s(T(x) - T(y))\rho(y)$$

and

$$\int \rho(x)\nabla\frac{\delta\mathcal{E}_1[\rho]}{\delta\rho}\cdot\big(x - T(x)\big) = c_{1,s}\int \rho(x)(x - T(x))k_s'(x-y)\rho(y)$$

(symmetrization)

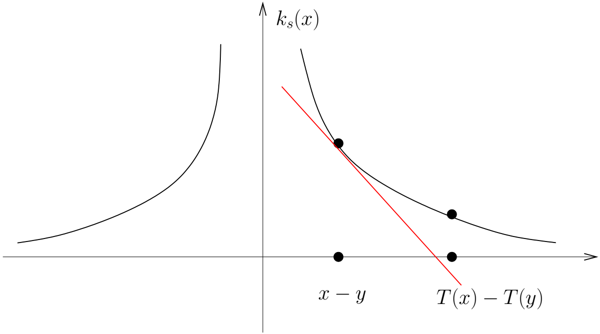
$$= \frac{c_{1,s}}{2}\int \rho(x)(x - T(x) - y + T(y))k_s'(x-y)\rho(y).$$

The final inequality with $\mathcal{E}_1[\rho] = \int \rho(-\Delta)^{-s}\rho$

We only have to prove that

$$k_s(x-y) - k_s\big(T(x)-T(y)\big) \le (x - T(x) - y + T(y))k_s'(x-y)$$

which is true because $x-y$ and $T(x)-T(y)$ have the same sign and $k_s(x)$ is convex on the two intervals $(-\infty, 0)$ and $(0, \infty)$ separately.



Regularization by diffusion

The formal computations are true for more general equations like

$$\rho_t = \nabla \cdot \left(\rho\left[\nabla(-\Delta)^{-s}\rho + \lambda x\right]\right) + \epsilon\rho_{xx}$$

with energy

$$\mathcal{E}_\epsilon[\rho] = \frac{1}{2}\int \rho(-\Delta)^{-s}\rho + \frac{\lambda}{2}\int |x|^2\rho + \epsilon\int(\log\rho - 1)\rho.$$

Exponential convergence

From the exponential convergence of $\mathcal{E}[\rho(t)] - \mathcal{E}[\rho_\infty]$:

- $W_2(\rho(t), \rho_\infty) \leq \sqrt{\frac{2}{\lambda}\left(\mathcal{E}[\rho(t)] - \mathcal{E}[\rho_\infty]\right)}$
- $\|(-\Delta)^{-s/2}(\rho(t) - \rho_\infty)\|_2^2 = \int(\rho(t) - \rho_\infty)(-\Delta)^{-s}(\rho(t) - \rho_\infty)$ $\leq 2\left(\mathcal{E}[\rho(t)] - \mathcal{E}[\rho_\infty]\right)$
- Provided a bound on higher regularity, for example $\|(-\Delta)^{s/2}(\rho(t) - \rho_\infty)\|_2$, we get exponential convergence in L^2,

$$\|\rho(t) - \rho_\infty\|_2 \leq \left(\|(-\Delta)^{-s/2}(\rho(t) - \rho_\infty)\|_2 \|(-\Delta)^{s/2}(\rho(t) - \rho_\infty)\|_2\right)^{1/2}$$

Conclusion and open problems

- The convergence in one dimension is exponential, in $\mathcal{E}[\rho] - \mathcal{E}[\rho_\infty]$, $W_2(\rho, \rho_\infty)$ and $\|(-\Delta)^{-s/2}(\rho - \rho_\infty)\|_2$, but not directly in other norms.
- The convergence rate in higher dimension is exponential is not known.
- The convergence for other related equations [7] [8] [9] ,

$$\rho_t = \nabla \cdot \left(\rho \nabla (-\Delta)^{-s} \rho^{m-1}\right), \quad \rho_t + (-\Delta)^s u^m = 0, \cdots$$

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