Sobolev and Hardy-Littlewood-Sobolev inequalities

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The sharp Sobolev and Hardy-Littlewood-Sobolev inequalities

Theorem (T. Aubin, 1976 and G. Talenti, 1976)

$$\mathcal{S}[u] := S_d \|\nabla u\|_2^2 - \|u\|^2_{\frac{2d}{d-2}} \geqslant 0\,, \tag{S}$$

where $S_d = \frac{1}{\pi\, d\, (d-2)} \left(\frac{\Gamma(d)}{\Gamma\left(\frac{d}{2}\right)} \right)^{\frac{2}{d}}$

Theorem (E.H. Lieb, 1983)

$$\mathcal{H}[v] := S_d \|v\|^2_{\frac{2d}{d+2}} - \int v\,(-\Delta)^{-1} v \geqslant 0\,. \tag{HLS}$$

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Both results include the equality case⋆, the Aubin-Talenti profiles:

$$\mathcal{S}[u] = \mathcal{H}[v] = 0 \;\text{ iff }\; u = u_* \,,\; v = v_* = u_*^{\frac{d+2}{d-2}} \,,\; \text{where } u_*(x) = (1+|x|^2)^{-\frac{d-2}{2}}$$

Legendre duality — E.H. Lieb, 1983

The Legendre transform F^* of a convex functional F is defined as

$$F^*[u] = \sup_v \left(\int u\, v - F[v] \right) .$$

Applying * to both sides of (S) one has

$$\left(u \mapsto \frac{1}{2} S_d \int |\nabla u|^2 \right)^* = v \mapsto \frac{1}{2} S_d^{-1} \int v (-\Delta)^{-1} v \,,$$

$$\left(u \mapsto \frac{1}{2} \|u\|^2_{\frac{2d}{d-2}} \right)^* = v \mapsto \frac{1}{2} \|v\|^2_{\frac{2d}{d+2}} \,.$$

And since $F_1 - F_2 \geqslant 0 \Rightarrow F_1^* - F_2^* \leqslant 0$, we get $\mathcal{S} \geqslant 0 \Leftrightarrow \mathcal{H} \geqslant 0$

Fast diffusion flow duality - I

For initial datum v_0 and $m<1$ one can consider the *fast diffusion equation*

$$\begin{cases} \frac{\partial}{\partial t} v = \Delta\left(v^m\right), & t>0,\ x\in\mathbb{R}^d \\ v(x,0)=v_0, \end{cases} \tag{FDE}$$

If v solves (FDE) with $v_0 \in L^{\frac{2d}{d+2}}$, $m=\frac{d-2}{d+2}$, then v is smooth and vanishes in finite time.

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If v solves (FDE) with $v_0 \in L^{\frac{2d}{d+2}}$, $m = \frac{d-2}{d+2}$, then v is smooth and vanishes in finite time. More precisely

Theorem (M. del Pino, M. Saez, 2001)

For any solution v to (FDE), there exists $T > 0$ such that, up to a fixed translation and scaling

$$v(t,x) = (\,T - t)^{\frac{d+2}{4}}\, v_*(x) + f\,(x,t)\,,$$

with $f \underset{t \to T}{\to} 0$.

Fast diffusion flow duality - II

Recall

$$\mathcal{S}[u] := S_d \|\nabla u\|_2^2 - \|u\|^2_{\frac{2d}{d-2}} \geqslant 0\,,$$
$$\mathcal{H}[v] := S_d \|v\|^2_{\frac{2d}{d+2}} - \int v(-\Delta)^{-1} v \geqslant 0\,.$$

The result of del Pino and Saez implies that

$$-\mathcal{H}[v(t)] \to 0\,,$$

and noticing

$$-\frac{1}{2}\mathcal{H}'[v(t)] = \left(\int v^{\frac{2d}{d+2}}\right)^{\frac{2}{d}} \mathcal{S}[v^{\frac{d-2}{d+2}}(t)]\,,$$

we get $\mathcal{S} \geqslant 0 \Rightarrow \mathcal{H}[v_0] \geqslant 0$ for any initial datum $v_0 \in L^{\frac{2d}{d+2}}$.

An improved Sobolev inequality using the flow

Theorem (J. Dolbeault, 2011)

For $d \geqslant 5$, there exists $\mathcal{C} > 0$ depending only on d such that

$$\mathcal{H}[u^{\frac{d+2}{d-2}}] \leqslant \mathcal{C}\, S_d \, \|u\|_{\frac{2d}{d-2}}^{\frac{8}{d-2}} \mathcal{S}[u] \qquad \text{(S-HLS)}$$

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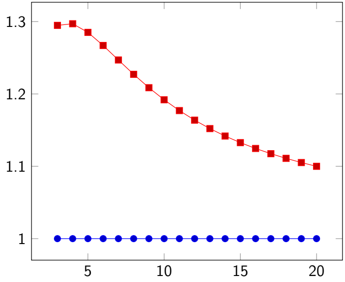

Improved Sobolev inequality, an alternative approach

Consider $(\mathrm{H}, \langle\cdot,\,\cdot\rangle)$ a Hilbert space and $\|u\|_p = \left\langle u^{\frac{p}{2}},\, u^{\frac{p}{2}}\right\rangle^{\frac{1}{p}}$. Suppose that $\mathcal{L}$ is a positive self-adjoint operator such that

$$\langle u,\, \mathcal{L}u\rangle \geqslant \|u\|_p^2\,,$$

then by Legendre transform we have

$$\left\langle v,\, \mathcal{L}^{-1}v\right\rangle \leqslant \|v\|_{p'}^2\,, \quad \frac{1}{p}+\frac{1}{p'}=1\,.$$

We can now expand the following square

$$0 \leqslant \|\mathcal{L}^{\frac{1}{2}}(\alpha u - \mathcal{L}^{-1}v)\|_2^2 = \alpha^2\,\langle u,\, \mathcal{L}u\rangle - 2\alpha\,\langle u,\, v\rangle + \left\langle v,\, \mathcal{L}^{-1}v\right\rangle\,.$$

$$0 \leqslant \alpha^2\,\langle u,\, \mathcal{L}u\rangle - \alpha\|u\|_p^p - \left(\alpha\|v\|_{p'}^{p'} - \left\langle v,\, \mathcal{L}^{-1}v\right\rangle\right)\,,$$

$$0 \leqslant \|u\|_p^{2(p-2)}\left(\langle u,\, \mathcal{L}u\rangle - \|u\|_p^2\right) - \left(\|v\|_{p'}^2 - \left\langle v,\, \mathcal{L}^{-1}v\right\rangle\right)\,.$$

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Take $\mathcal{L} = -S_d\,\Delta$ and get: $\mathcal{H}[u^{\frac{d+2}{d-2}}] \leqslant S_d \|u\|_{\frac{2d}{d-2}}^{\frac{8}{d-2}}\,\mathcal{S}[u]\,.$

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For $d \geqslant 3$, there exists $\mathcal{C} > 0$ depending only on d such that

$$\mathcal{H}[u^{\frac{d+2}{d-2}}] \leqslant \mathcal{C}\, S_d \, \|u\|_{\frac{2d}{d-2}}^{\frac{8}{d-2}} \, \mathcal{S}[u]\,, \qquad \text{(S-HLS)}$$

with $\frac{d}{d+4} \leqslant \mathcal{C} < 1$.

Lower bound on C_d

$$\frac{1}{\mathcal{C}\,S_d} = \inf_u \frac{\|u\|^{\frac{8}{d-2}}_{\frac{2d}{d-2}}\,\mathcal{S}[u]}{\mathcal{H}[u^q]} =: \inf_u \mathcal{Q}[u]\,, \qquad \mathcal{S}[u_*] = \mathcal{H}[u_*^q] = 0\,.$$

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take $u_\epsilon = u_* + \epsilon f$: $\quad \displaystyle \frac{1}{\mathcal{C}S_d} \leqslant \inf_f \lim_{\epsilon\to 0} \frac{\|u_\epsilon\|_{\frac{2d}{d-2}}^{\frac{8}{d-2}} \mathcal{S}[u_\epsilon]}{\mathcal{H}[u_\epsilon^q]} = \inf_{f \notin \ker LH} \frac{\|u_*\|_{\frac{2d}{d-2}}^{\frac{8}{d-2}} LS[f]}{LH[f]}$

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$$LS[f] = S_d \left(\|\nabla f\|_2^2 - d(d+2) \int \frac{f^2}{(1+|x|^2)^2} \right)$$

$$LH[f] = \left(\tfrac{d+2}{d-2}\right)^2 \left(\frac{1}{d\,(d+2)} \int \frac{f^2}{(1+|x|^2)^2} - \int \frac{f}{(1+|x|^2)^2} (-\Delta)^{-1} \frac{f}{(1+|x|^2)^2} \right)$$

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Using the eigenfunctions of the Laplace-Beltrami operator on the sphere, we get

$$\frac{1}{\mathcal{C}} \leqslant \inf_{f \notin \ker LH} \quad \frac{LS[f]}{d^2 (d+2)^2 LH[f]} = \frac{d+4}{d}$$

A non linear improved inequality and strict upper bound - I

Previously:

$$\mathsf{J} = \int v^{\frac{2d}{d+2}}\,, \quad \mathsf{J}' = -(m+1)\int |\nabla v^m|^2\,, \quad -\frac{1}{2}\mathcal{H}'[v(t)] = \mathsf{J}^{\frac{2}{d}}\,\mathcal{S}[v^{\frac{d-2}{d+2}}(t)]\,.$$

Then there exists Y s.t. $\mathcal{H}(t) = \mathsf{Y}(\mathsf{J}(t))$

$$-\mathsf{Y}'(\mathsf{J}(t))\mathsf{J}'(t) = \mathcal{H}'(t) \leqslant -\frac{\mathcal{H}_0'}{\mathsf{J}_0}\,\mathsf{J}$$

By writing $\mathcal{C} = \mathsf{C}_d/\mathsf{S}_d$ and using the improved Sobolev's inequality

$$\frac{1}{2}\mathsf{S}_d^{-1}\,\mathsf{J}^{-(1+\frac{2}{d})}\mathcal{H}_0^2 + \mathcal{C}\,\mathcal{H}_0 \leqslant -\frac{1}{2}\mathcal{C}\,\mathcal{H}_0'\,\mathsf{S}_d\,\mathsf{J}_0^{\frac{2}{d}} = \mathcal{C}\,\mathsf{S}_d\,\mathsf{J}_0^{\frac{4}{d}}\,\mathcal{S}_0\,,$$

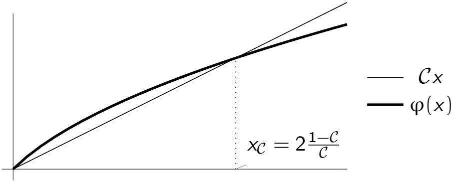
and then

$$\mathcal{H}_0 \leqslant \mathsf{S}_d\mathsf{J}_0^{1+\frac{2}{d}}\left(\sqrt{\mathcal{C}^2 + 2\,\mathcal{C}\,\mathsf{J}_0^{1-\frac{2}{d}}\mathcal{S}_0} - \mathcal{C}\right)$$

A non linear improved inequality and strict upper bound - II

With $\varphi = \sqrt{\mathcal{C}^2 + 2\mathcal{C}x} - \mathcal{C}$,

$$\mathcal{H} \leqslant S_d \|u\|_{\frac{2d}{d-2}}^{\frac{d+2}{d}} \varphi\left(\|u\|_{\frac{2d}{d-2}}^{\frac{d-2}{d}} \mathcal{S}\right) .$$



Assume $\mathcal{C} = 1$, $\mathrm{J} = \mathrm{J}_*$ and consider (u_n) a minimizing seq. of $\mathcal{Q}[u] = \frac{\mathcal{S}[u]}{\mathcal{H}[u^q]}$.

- If $\lim \mathcal{S}[u_n] > 0$ and $\lim \mathcal{H}[u_n^q] > 0$, then

$$\begin{aligned} 0 &= \lim \mathrm{S}_d \mathrm{J}_*^{\frac{4}{d}} \mathcal{S} - \mathcal{H} \\ &= \mathrm{S}_d \mathrm{J}_*^{1+\frac{2}{d}} \left(\lim \mathrm{J}_*^{\frac{2}{d}-1} \mathcal{S} - \varphi\left(J_*^{\frac{2}{d}-1} \mathcal{S}\right)\right) \\ &\quad + \mathrm{S}_d \left(\lim \mathrm{J}_*^{1+\frac{2}{d}} \varphi\left(J_*^{\frac{2}{d}-1} \mathcal{S}\right) - \mathcal{H}\right) \end{aligned}$$

- If $\lim \mathcal{S}[u_n] = \lim \mathcal{H}[u_n^q] = 0$, then $\lim \mathcal{Q}[u_n]$ is given by the linearization. This contradicts $C_d = S_d$.

Extensions

Theorem (Caffarelli-Kohn-Nirenberg inequalities - J. Dolbeault, G. J., 2013)

$$\left(\int \frac{|v|^q}{|x|^{(2a-b)\,q}}\right)^{\frac{2}{q}} - \frac{1}{C_{a,b}}\,\langle v, \mathsf{L}_a^{-1}\, v\rangle \leqslant C_{a,b} \int \frac{|\nabla u|^2}{|x|^{2a}} - \left(\int \frac{|u|^p}{|x|^{bp}}\right)^{\frac{2}{p}}$$

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Theorem (Fractional Laplace operator - G. J., V.H. Nguyen, 2014)

For any $d \geqslant 2$, and $0 < s < \frac{d}{2}$, denote $q = \frac{d+2s}{d-2s}$.

$$S_{d,s} \|u^q\|^2_{\frac{2d}{d+2s}} - \int_{\mathbb{R}^d} u^q\,(-\Delta)^{-s} u^q \,dx \leqslant \mathcal{C}\, S_{d,s}\, \|u\|^{\frac{8s}{d-2s}}_{\frac{2d}{d-2s}} \left(S_{d,s}\|u\|_s^2 - \|u\|^2_{\frac{2d}{d-2s}}\right)$$

The best constant $\mathcal{C}$ is such that

$$\frac{d-2s+2}{d+2s+2} \leqslant \mathcal{C} \leqslant 1\,,$$

The last inequality is strict if $0 < s < 1$.

$$S_{d,s}\|u^q\|^2_{\frac{2d}{d+2s}} - \int_{\mathbb{R}^d} u^q\,(-\Delta)^{-s} u^q\;dx \leqslant \mathcal{C}\, S_{d,s}\;\|u\|^{\frac{8s}{d-2s}}_{\frac{2d}{d-2s}}\left(S_{d,s}\|u\|_s^2 - \|u\|^2_{\frac{2d}{d-2s}}\right)\,,$$

with

$$u = \left(1 + \frac{d-2s}{2d}\,V(\Sigma^{-1}(x))\right) J_{\Sigma^{-1}}(x)^{-\left(s-\frac{d}{2}\right)}\,,$$

Write it on the sphere $\mathbb{S}^d$ in terms of V,

$$S_{d,s}\|u^q\|^2_{\frac{2d}{d+2s}}-\int_{\mathbb{R}^d}u^q\,(-\Delta)^{-s}u^q\,dx\leqslant\mathcal{C}\,S_{d,s}\,\|u\|^{\frac{8s}{d-2s}}_{\frac{2d}{d-2s}}\left(S_{d,s}\|u\|_s^2-\|u\|^2_{\frac{2d}{d-2s}}\right),$$

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Write it on the sphere $\mathbb{S}^d$ in terms of V, [...], and take $s\to\frac{d}{2}^-$

Theorem (G. J., V.H. Nguyen, 2014)

$$\mathrm{Ent}_\sigma(e^F)+\frac{d}{M}\iint_{\mathbb{S}^d\times\mathbb{S}^d}e^{F(\xi)}\,\log|\xi-\eta|\,e^{F(\eta)}\,d\sigma(\xi)\,d\sigma(\eta)$$
$$+M\frac{d}{2}\left(\Psi(d)-\Psi\left(\frac{d}{2}\right)-\log 4\right)$$
$$\leqslant\mathcal{C}\,M\left[\frac{1}{2d}\sum_{k\geqslant1}\frac{\Gamma(k+d)}{\Gamma(d)\Gamma(k)}\int_{\mathbb{S}^d}|F_k|^2d\sigma+\int_{\mathbb{S}^d}F\,d\sigma-\log M\right]$$

$d = 2$: Onofri's inequality

Theorem (J. Dolbeault, G. J., 2013)

Assume that $d = 2$. The inequality

$$\int_{\mathbb{R}^2} g \log\left(\frac{g}{M}\right) - \frac{4\pi}{M}\int_{\mathbb{R}^2} g\,(-\Delta)^{-1} g + M\,(1+\log\pi) \leqslant C_2\,M\left[\frac{1}{16\,\pi}\|\nabla f\|_2^2 + \int_{\mathbb{R}^2} f\,d\mu - \log M\right] \quad (1)$$

holds for any function $f \in \mathcal{D}(\mathbb{R}^2)$ such that $M = \int_{\mathbb{R}^2} e^f\,d\mu$ and $g = e^f\,\mu$.

Theorem (G. J., V.H. Nguyen, 2014)

$$\frac{1}{3} \leqslant C_2 \leqslant 1$$

Thanks!