Approximating parabolic PDEs with oblique boundary data

Inwon Kim
UCLA (joint work with Damon Alexander and Minha Yoo)

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Model Problem

For a bounded domain $\Omega \subset \mathbb{R}^n$ with C^2 boundary, we consider the Neumann problem

$$\begin{cases} u_t - \Delta u = 0 & \text{in} \quad \Omega \times (0,\infty); \\ Du \cdot \nu = 0 & \text{on} \quad \partial\Omega \times [0,\infty); \\ u(x,0) = u_0(x) & \text{in} \quad \Omega. \end{cases}$$

Can we approximate $u\chi_\Omega$ with

$$\begin{cases} v_t - \Delta v - N\nabla \cdot (v\vec{\beta}) = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ v(x,0) = u_0\chi_\Omega \end{cases}$$

where $\vec{\beta} \sim \nu$ in $\mathbb{R}^n - \Omega$ and $\vec{\beta} = 0$ in Ω (e.g. $\vec{\beta} = \nabla[d(x,\Omega)]^2$)?

What can we say about general velocity field $\vec{\beta}$ which points inward to Ω?

The general problem

Let us consider

$$(P_g) \qquad \begin{cases} u_t - F(D^2u, Du, u, x, t) = 0 & \text{in} \quad \Omega \times (0,\infty); \\ Du \cdot \vec{l}(x,t) = 0 & \text{on} \quad \partial\Omega \times [0,\infty); \\ u(x,0) = u_0(x) & \text{in} \quad \Omega. \end{cases}$$

Here, F is a quasilinear, uniformly elliptic operator

$$F(D^2u, Du, u, x, t) = \sum_{i,j} a_{ij}(u, x, t)u_{x_i x_j} + b(Du, u, x, t).$$

and $\vec{l} : \mathbb{R}^n \times (0, \infty) \to \mathbb{R}^n$, a smooth vector field, satisfies the oblique condition $\vec{l} \cdot \nu > c_0 > 0$ on $\partial\Omega$, where $\nu = \nu_x$ is the outward normal of Ω.

Due to the oblique condition, we can show that $\vec{v}$ can be written as

$$\vec{l}(x, t) = A(x, t)\vec{\nu}(x),$$

where A is symmetric and uniformly positive definite.

The approximating problem

- Now consider $\Phi(x) = d^2(x, \Omega)$ near $\partial\Omega$ and consider

$$(P_N) \quad \begin{cases} v_t - F^*(D^2v, Dv, v, x, t) - N\nabla \cdot [vA(x,t)\nabla\Phi] = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } \mathbb{R}^n \times (0, \infty); \\ v(x, 0) = v_0(x) \text{ in } \mathbb{R}^n. \end{cases}$$

- Here $v_0 = u_0\chi_\Omega$ and
-

$$F^*(D^2v, Dv, v, x, t) = \begin{cases} F(D^2v, Dv, v, x, t) & \text{if } x \in \Omega \\ \nabla \cdot (A(x,t)\nabla v) & \text{if } x \in \Omega^c. \end{cases}$$

In the case $F = \nabla \cdot (A \nabla v)$, v solves the Fokker-Planck equation

$$v_t - \nabla \cdot (A \nabla v) - N \nabla \cdot [v A \nabla \Phi] = 0,$$

which represents the probability density of the particles in $\mathbb{R}^n$ evolving with

$$dX_t = N(-A \nabla \Phi) dt + S dW_t,$$

where $A = SS^T$. Hence if the drift $-(A \nabla \Phi)$ points toward Ω and vanishes on $\partial\Omega$ then one expects v converges to a compactly supported function $u \chi_\Omega$ as $N \to \infty$. For our choice of $\Phi = d^2$, $\nabla \Phi$ is parallel to ν and thus we expect to obtain the co-normal boundary data $\nabla u \cdot A\nu = 0$.

The main theorem

Theorem

Let u and v respectively solve (P_g) and (P_N) as given above. Then for any $T > 0$, v_N uniformly converges to $u\chi_\Omega$ in $[(\mathbb{R}^n - \Omega) \cup \bar{\Omega}] \times [0, T]$ as $N \to \infty$. Moreover we have

$$|v_N(x,t) - u(x,t)| \leq CN^{-1/2} \text{ in } \bar{\Omega} \times [0, T], \tag{1}$$

and $|v_N| \leq N^{1/2} e^{-Nd}$ outside of Ω. Here C depends only on n, λ, Λ, T and the regularity of the coefficients and the domain Ω.

Literature

Previous results concern the Kolmogorov Backward equation

$$w_t - F(D^2 w, Dw, w, x, t) + NDw \cdot \vec{\beta} = 0$$

associated with the SDE

$$dX_t = N(-\vec{\beta})dt + SdW_t,$$

and its convergence to the oblique boundary problem.

- Menaldi (1983), Lions-Snitzman (1984), Cepa (1998), ... in the setting of SDEs.
- In the PDE setting: for elliptic (nonlocal) case: Barles-Georgelin-Jakobsen (2013).

In these cases w converges to u in Ω and $u(\pi(x))$ outside of Ω as $N \to \infty$, where $\pi(x)$ is generated by the characteristic ODE $\dot{\pi}_x(t) = \vec{\beta}(\pi(t))$ near $\partial\Omega$.

In the parabolic case this necessitates choosing the initial data for the approximating equation w with a priori information on the true solution u...

Main observation

- If $F(D^2u) = \nabla \cdot (A\nabla u)$, then $w := e^{N\Phi} v$ solves

$$w_t - \nabla \cdot (A\nabla u) - NDw \cdot (A\nabla\Phi) = 0.$$

- Now we can construct barriers (super and sub-solutions) that are based on the true solution u for w in $\mathbb{R}^n \times [0, \infty)$ to compare u and $ve^{-N\Phi}$ and obtain the error estimate.

Patching up diffusion operators

Using the error estimate obtained above, we now are able to patch up the general quasi-linear operator with the linear diffusion operator outside of the domain Ω to obtain the main result. Here we use the regularity results by Kim-Krylov (2007, 2009) for operators with discontinuous coefficients with respect to one direction in $\mathbb{R}^n$.

Back to the Model problem

- We consider the heat equation with Neumann data over an interval $[a, b] := \Omega$:

$$(H) \qquad \begin{cases} u_t - u_{xx} = 0 & \text{in } \Omega \times (0, \infty); \\ u_x = 0 & \text{at } \{a, b\} \times (0, \infty); \\ u(x, 0) = u_0(x) & \text{in } [a, b]. \end{cases}$$

- The approximating problem is

$$v_t - v_{xx} - N(v_x \Phi_x + v \Phi_{xx}) = 0 \text{ in } \Omega \times (0, \infty);$$
$$v(x, 0) = v_0(x) \text{ in } \Omega.$$

- Here

$$v_0(x) := \begin{cases} u_0(x) & \text{if } x \in [a, b] \\ u_0(b) e^{-N\Phi(x)} & \text{if } x > b \\ u_0(a) e^{-N\Phi(x)} & \text{if } x < a. \end{cases}$$

We wish to create a supersolution to extend u off Ω of the form

$$\varphi(x,t) = f(x,t)e^{-N\Phi(x)}.$$

This is a supersolution provided

$$f_t - f_{xx} + N f_x \Phi_x \geq 0.$$

Let $\epsilon := 10N^{1/2}$ and consider perturbing u by $u_\epsilon \sim u + O(\epsilon)$ such that $u_x = \epsilon > 0$ on $\partial\Omega$, and then we construct f to the right of b so that it matches u_ϵ at the boundary:

$$f(x,t) := u_\epsilon(b,t) + \alpha\frac{(x-b)^2}{\epsilon} + \alpha\epsilon(x-b).$$

Then we get a final supersolution by writing

$$w(x,t) = \begin{cases} \tilde{\varphi} & \text{if } x \leq a \\ u_\epsilon(x,t) & \text{if } a < x < b \\ \varphi(x,t) & \text{if } x \geq b. \end{cases}$$

This starts above v_0 by construction, so by comparison principle $v \leq w$ in $\mathbb{R} \times [0,\infty)$.

Model problem: conclusion

- Similarly one can create a subsolution, and this lets us bound

$$u_{-\epsilon}(x,t) \le v(x,t) \le u_{\epsilon}(x,t),$$

  in $[a,b]$. Then the definition of u_ϵ gives us the bound

$$\|u - v\|_{L^\infty(\Omega\times[0,T])} < C(u_0,a,b)(T+1)N^{-1/2}.$$

- We remark that by comparison with above barriers the natural choice of initial data $v_0(x) = u_0\chi_\Omega$ can also sandwiched by these barriers to give convergence with the same rate.

The general case

- The quasi-linear case works by approximating the problem with the linear divergence form operator near the boundary. Define

$$\Omega_r := \{x \in \Omega : d(x, \partial\Omega) > r\}.$$

- Then we define

$$F_r(D^2u, Du, u, x, t) := \\ g(x)F(D^2u, Du, u, x, t) + (1 - g(x))\nabla \cdot (A(x,t)\nabla u)$$

- g is defined to smoothly interpolate between F in Ω_{2r} and $\nabla \cdot (A(x,t)\nabla u)$ outside of Ω_r.

The auxiliary problems

- Our first auxiliary problem is a modification of the original problem with the interpolating operator:

$$(P_r) \quad \begin{cases} w_t - F_r(D^2 w, Dw, w, x, t) = 0 & \text{in} \quad \Omega \times [0, T]; \\ (A(x,t)\nabla w) \cdot \vec{\nu} = 0 & \text{on} \quad \partial\Omega \times [0, T]; \\ w(x, 0) = u_0(x) & \text{in} \quad \Omega. \end{cases}$$

- The second is the corresponding modification of the approximating problem:

$$(P_{r,N}) \quad \begin{cases} \tilde{v}_t - F_r(D^2\tilde{v}, D\tilde{v}, \tilde{v}, x, t) - N\nabla \cdot [\tilde{v}A(x,t)\nabla\Phi] = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{in } \mathbb{R}^n \times [0, \infty); \\ \tilde{v}(x, 0) = v_0(x) \text{ in } \mathbb{R}^n. \end{cases}$$

The ingredients

- The first step comes from a theorem of Kim-Krylov (2007), which gives uniform $W^{2,p}$ estimates on w as $r \to 0$ for all $p > 0$ and as the operator becomes discontinuous in the normal direction.
- This along with the barrier arguments help us get uniform $C_x^{1,1}$ and Lipschitz-in-time estimates on w up to $\partial\Omega$, independently of r.
- Then the convergence of $\tilde{v}$ to w, uniformly in r, follows since the extension only relied on these estimates on the true solution w on $\partial\Omega$.

The last convergence results

- Next we show that the two auxiliary problems converge as we take $r \to 0$. Showing $w \to u$ as $r \to 0$ follows by building a barrier function to patch u in $\Omega \backslash \Omega_{2r}$ in a way that lets us control the behavior of w.
- Lastly, we show the convergence of $\tilde{v}$ to v as $r \to 0$. This follows by showing $\tilde{v}$ has a subsequential limit and this limit must be a viscosity solution.

Does the diffusion operator need to change outside of the domain?

Suppose we have

$$v_t - \Delta v - N\nabla \cdot (vd\vec{\beta}) = 0 \quad \text{in } \mathbb{R}^n \times (0,\infty).$$

with initial data v_0 supported in Ω, where and $\vec{\beta} \cdot \vec{\nu} > 0$ but $\vec{\beta} \neq Dd$. Then do v converge to $u\chi_\Omega$ with $u(x,0) = v_0$, where u is a solution to a boundary value problem for the heat equation?

Our guess (in progress) is that the answer is yes if and only if $\vec{\beta} \cdot \nu_x \equiv C > 0$ on $\partial\Omega$. Boundary condition appear to depend on the tangential component of $\vec{\beta}$.

More general nonlinearities?

$$v_t - F(D^2 v, Dv, x, t) + N\nabla \cdot (vd\vec{\beta}) = 0?$$