A thin film approximation of the Muskat problem

Philippe Laurençot

CNRS, Institut de Mathématiques de Toulouse

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Joint works with:

Bogdan-Vasile Matioc (Hannover)

TOULOUSE

The Muskat problem I

Model for the motion of two immiscible fluids with different densities and viscosities in a porous medium (intrusion of water into oil).

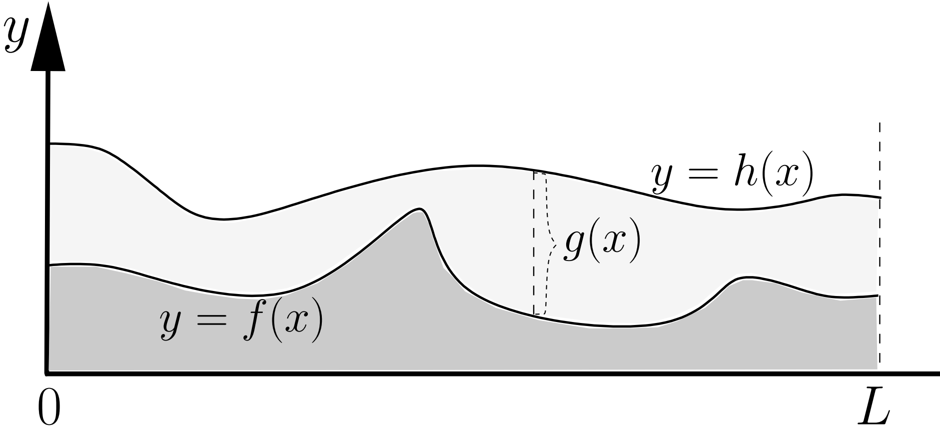
- Bottom of the porous medium: $y = 0$,
- Height of the lower fluid: $y = f(t, x)$, $\Gamma(f) := \{y = f\}$
- Domain occupied by the lower fluid:

$$\Omega(f) := \{(x, y) \in (0, L) \times (0, \infty) \ : \ y < f(t, x)\},$$

- Height of the upper fluid: $y = h(t, x)$, $\Gamma(h) := \{y = h\}$
- Domain occupied by the upper fluid:

$$\Omega(f, h) := \{(x, y) \in (0, L) \times (0, \infty) \ : \ f(t, x) < y < h(t, x)\}.$$

The Muskat problem II



The Muskat problem III

$$\begin{array}{lcll}
\Delta u_+ & = & 0 & \text{in} \quad \Omega(f,g), \\
\Delta u_- & = & 0 & \text{in} \quad \Omega(f), \\
\partial_t h & = & -\mu_+^{-1} \langle \nabla u_+, (-\partial_x h, 1) \rangle & \text{on} \quad \Gamma(h), \\
u_+ & = & G\rho_+ h - \gamma_d \kappa_{\Gamma(h)} & \text{on} \quad \Gamma(h), \\
\partial_t f & = & -\mu_\pm^{-1} \langle \nabla u_\pm, (-\partial_x f, 1) \rangle & \text{on} \quad \Gamma(f), \\
u_+ - u_- & = & G(\rho_+ - \rho_-) f + \gamma_w \kappa_{\Gamma(f)} & \text{on} \quad \Gamma(f), \\
\partial_y u_- & = & 0 & \text{on} \quad \{y = 0\}.
\end{array}$$

with initial data $0 < f_0 < h_0$ and

- $\rho_\pm, \mu_\pm$: density and viscosity of the fluid $\pm$,
- G: gravity constant,
- $u_\pm = p_\pm + G\rho_\pm y$, $\mathbf{v}_\pm = -\mu_\pm^{-1} \nabla u_\pm$ (Darcy's law),
- $\gamma_w, \kappa_{\Gamma(f)}$: surface tension and curvature of the interface $\Gamma(f)$,
- $\gamma_d, \kappa_{\Gamma(h)}$: surface tension and curvature of the interface $\Gamma(h)$.

Thin fluid layers: $h \ll L$

Scaling:

$$x = \tilde{x}, \quad y = \varepsilon \tilde{y}, \quad f = \varepsilon \tilde{f}, \quad h = \varepsilon \tilde{h}, \quad u_\pm = \tilde{u}_\pm, \quad t = \tilde{t}/\varepsilon.$$

Formal asymptotic expansion in powers of ε:

$$\begin{aligned}
\partial_t f &= \mu_-^{-1} G(\rho_- - \rho_+)\partial_x(f\partial_x f) + \mu_-^{-1} G\rho_+ \partial_x(f\partial_x h) \\
&\quad -\mu_-^{-1}\gamma_w \partial_x(f\partial_x^3 f) - \mu_-^{-1}\gamma_d \partial_x(f\partial_x^3 h) \\
\partial_t h &= \mu_-^{-1} G(\rho_- - \rho_+)\partial_x(f\partial_x f) + \mu_-^{-1} G\rho_+ \partial_x(f\partial_x h) \\
&\quad +\mu_+^{-1} G\rho_+ \partial_x((h-f)\partial_x h) - \mu_+^{-1}\gamma_d \partial_x((h-f)\partial_x^3 h) \\
&\quad -\mu_-^{-1}\gamma_w \partial_x(f\partial_x^3 f) - \mu_-^{-1}\gamma_d \partial_x(f\partial_x^3 h)
\end{aligned}$$

[Escher, Matioc & Matioc (2012)]

Thin film system

Neglecting the curvature terms ($\gamma_w = \gamma_d = 0$) and rescaling time give:

$$\begin{aligned} \partial_t f &= \partial_x(f\partial_x f) + R\partial_x(f\partial_x h), \\ \partial_t h &= \partial_x(f\partial_x f) + R\partial_x(f\partial_x h) + R_\mu\partial_x((h-f)\partial_x h), \end{aligned}$$

with initial conditions $0 < f_0 < h_0$, where

$$R := \frac{\rho_+}{\rho_- - \rho_+} > 0 \quad \text{and } R_\mu := \frac{\mu_-}{\mu_+} R > 0.$$

[Escher, Matioc & Matioc (2012)]

Remark. If $f \equiv 0$, reduction to the PME $\partial_t h = \partial_x(h\partial_x h)$.

Alternative formulation

Define $g := h - f \geq 0$. Then a rescaled version of (f, g) solves

$$\begin{aligned} \partial_t f &= \partial_x \left[f \, \partial_x \left((1+R)\eta^2 f + Rg \right) \right] , \\ \partial_t g &= R_\mu \partial_x \left[g \, \partial_x \left(\eta^2 f + g \right) \right] , \end{aligned}$$

in $(0, \infty) \times \mathbb{R}$ with initial conditions $f_0 \geq 0$ and $g_0 \geq 0$ satisfying

$$\|f_0\|_1 = \|g_0\|_1 = 1 ,$$

and $R > 0$, $\eta > 0$, and $R_\mu > 0$.

Degenerate parabolic system with full diffusion matrix

Properties

- $f \geq 0$ and $g \geq 0$ by the comparison principle,
- $\|f(t)\|_1 = 1$ and $\|g(t)\|_1 = 1$,
- Energy functional: $\mathcal{E}_2(f,g) := \left(\eta^2\|f\|_2^2 + R\|\eta f + \eta^{-1} g\|_2^2\right)/2$ with

$$\frac{d}{dt}\mathcal{E}_2(f,g) = -\left\|\sqrt{f}\,\partial_x\left(\eta^2(1+R)f + Rg\right)\right\|_2^2 - \frac{RR_\mu}{\eta^2}\left\|\sqrt{g}\,\partial_x\left(\eta^2 f + g\right)\right\|_2^2 \leq 0.$$

- Entropy functional:

$$\mathcal{E}_1(f,g) := \|f\ln f - f + 1)\|_1 + \frac{R}{R_\mu}\|g\ln g - g + 1\|_1$$

with

$$\frac{d}{dt}\mathcal{E}_1(f,g) = -\eta^2\|\partial_x f\|_2^2 - \frac{R}{\eta^2}\|\partial_x(\eta^2 f + g)\|_2^2 \leq 0.$$

Variational approach

$$\partial_t f = \partial_x \left[f\, \partial_x \left(\eta^2 (1+R) f + R g \right) \right] = \partial_x \left[f\, \partial_x \left(\frac{\partial \mathcal{E}_2}{\partial f} \right) \right] ,$$

$$\partial_t g = R_\mu \partial_x \left[g\, \partial_x \left(\eta^2 f + g \right) \right] = \frac{\eta^2 R_\mu}{R} \partial_x \left[g\, \partial_x \left(\frac{\partial \mathcal{E}_2}{\partial g} \right) \right] ,$$

with initial conditions $f_0 \in \mathcal{P}(\mathbb{R})$ and $g_0 \in \mathcal{P}(\mathbb{R})$.

Energy functional: $\mathcal{E}_2(f,g) := \left(\eta^2 \|f\|_2^2 + R \|\eta f + \eta^{-1} g\|_2^2 \right) /2$

Gradient flow of $\mathcal{E}_2$ with respect to the 2-Wasserstein distance W_2 in $\mathcal{P}_2(\mathbb{R}) \times \mathcal{P}_2(\mathbb{R})$.

Existence

- Weak solutions by a variational approach: $f_0, g_0 \in \mathcal{P}_2(\mathbb{R}) \cap L^2(\mathbb{R})$. [L. & Matioc (2013)]
- Improved regularity of the minimisers. [Matthes, McCann & Savaré (2009)]
- Uniqueness of weak solutions?
- Gradient flow: the full system with capillarity fits as well in this framework, in $\mathbb{R}$ and $\mathbb{R}^2$ [L. & Matioc].

Related results

- Weak solutions by a compactness approach (in a bounded interval). Global existence, asymptotic stability. Also for the full system with capillarity. [Escher, L. & Matioc (2011)], [Matioc (2012)]
- Strong solutions: $f_0, g_0 \in H^2_N(0, L)$, $f_0 > 0$, $g_0 > 0$. Local existence, uniqueness, local asymptotic stability. Also for the full system with capillarity. [Escher, Matioc & Matioc (2012)], [Escher & Matioc (2013)]
- Convergence of solutions to Muskat problem towards a solution of the "thin film" approximation [Matioc & Prokert (2012)].

Large time behaviour

The porous medium equation I

$$\begin{aligned} \partial_t w &= \partial_x^2 \left(w^2 \right) , \quad (t,x) \in (0,\infty) \times \mathbb{R} , \\ w(0) &= w_0 , \quad x \in \mathbb{R} , \end{aligned}$$

with $w_0 \in L^1(\mathbb{R})$, $w_0 \geq 0$.

- Scale invariance: if w is a solution then so are

$$(t,x) \mapsto \lambda w(\lambda^3 t, \lambda x) .$$

- Solutions invariant with respect to this transformation: self-similar solutions.

$$(t,x) \mapsto t^{-1/3} \mathcal{B}_M \left(x t^{-1/3} \right) , \quad \mathcal{B}_M(x) = \left(C(M) - \frac{x^2}{12} \right)_+$$

with $C(M) > 0$ such that $\|\mathcal{B}_M\|_1 = M$.

The porous medium equation II

- Explicit solutions (Barenblatt solutions):

$$\mathcal{B}_M(x) = \left(C(M) - \frac{x^2}{12} \right)_+ , \quad \|\mathcal{B}_M\|_1 = M .$$

- Attractors for the dynamics: if $\|w_0\|_1 = M$,

$$w(t,x) \sim t^{-1/3} \mathcal{B}_M \left(x t^{-1/3} \right) \quad \text{as} \quad t \to \infty .$$

- Decay rates.

Large time behaviour in $\mathbb{R}$

- same scaling invariance as the PME
- existence of self-similar solutions.

We look for solutions of the form

$$t^{-1/3}(F,G)\left(xt^{-1/3}\right) \ .$$

Self-similar solutions

Rescaling + self-similar variables: the profiles (F, G) are stationary solutions to

$$\partial_t f = \partial_x \left[f\, \partial_x \left((1+R)\, \eta^2\, f + R\, g + \frac{x^2}{6} \right) \right] ,$$
$$\partial_t g = \partial_x \left[g\, \partial_x \left(R_\mu\, \eta^2\, f + R_\mu\, g + \frac{x^2}{6} \right) \right] .$$

such that $F, G \geq 0$ and $\|F\|_1 = \|G\|_1 = 1$.

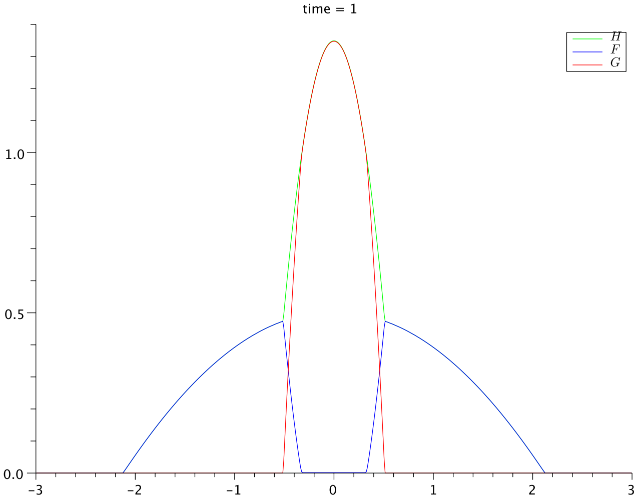
The situation is more complicated than for the PME equation [L. & Matioc].

Self-similar profiles

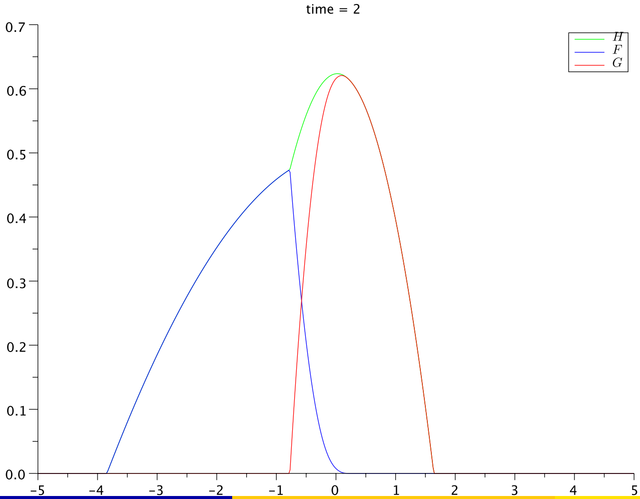
Given R, R_μ, and η:

- Existence of a unique even solution (F_0, G_0).
- Existence of a continuum $(F_\lambda, G_\lambda)_{\lambda\in\Lambda}$ of solutions if $R_\mu \notin (R_\mu^-, R_\mu^+)$. Here Λ is a closed interval, $0 \in \Lambda$, (F_λ, G_λ) are non-symmetric if $\lambda \neq 0$, and either F_λ or G_λ has a disconnected support if λ lies in the interior of Λ, $\lambda \neq 0$.
- If λ is an end point of Λ, then F_λ and G_λ have connected supports if $R_\mu^+ < R_\mu < R_\mu^M$ or $R_\mu^m < R_\mu < R_\mu^-$.
- Attractors for the dynamics. Decay rates?

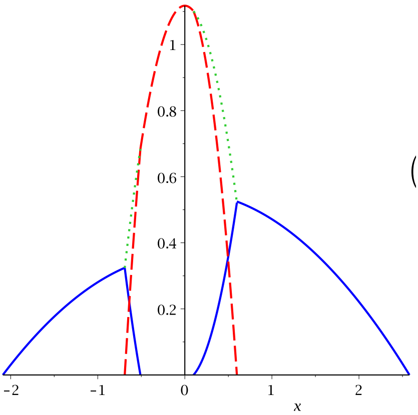
Symmetric self-similar profile

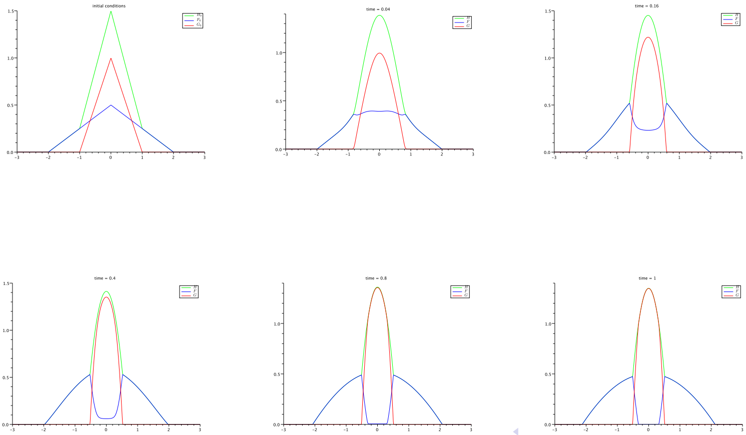


Non-symmetric self-similar profile with connected supports



Asymmetric self-similar profile with disconnected supports





initial conditions
time = 0.04
time = 0.16
time = 0.4
time = 0.8
time = 1

Evolution in self-similar variables