The Curvature-Dimension Condition with Finite N

—

Consequences and Transformations

Karl-Theodor Sturm

Hausdorff Center for Mathematics
Universität Bonn

Outline

- Two Possible Introductions
- Equivalence of Bakry-Emery and Curvature-Dimension Conditions
- Gradient Flows for (K, N)-Convex Functions
- Time Change and Conformal Transformation
- Analysis and Geometry on Metric Measure Spaces

Two Possible Introductions

Heat flow on (X, d, m) is the gradient flow for the entropy S

- Functional inqualities are determind by convexity bounds for S (= Ricci bounds for X): $CD(K, \infty)$-condition
- What do we gain if we assume the more restrictive $CD(K, N)$-condition with finite N ?
- What can we say about the transformed space (X, d', m') with $m' = e^V m$ and $d' \approx e^W d$ on infinitesimal scale?

Two Possible Introductions

Heat flow on (X, d, m) is the gradient flow for the entropy S

- Functional inqualities are determind by convexity bounds for S ($=$ Ricci bounds for X): $CD(K, \infty)$-condition
- What do we gain if we assume the more restrictive $CD(K, N)$-condition with finite N ?
- What can we say about the transformed space (X, d', m') with $m' = e^V m$ and $d' \approx e^W d$ on infinitesimal scale?

Heat flow is defined by the diffusion operator L on $L^2(X, m)$

- Functional inqualities are determind by the Bakry-Emery condition $BE(K, \infty)$ for L
- What do we gain if we assume the more restrictive $BE(K, N)$-condition with finite N ?
- What can we say about the transformed operators $\tilde{L}u = Lu + \Gamma(V, u)$ or $L'u = e^{-2W} Lu$?

L^2-Wasserstein Space for Riemannian M

Given $\mu_0, \mu_1 \in \mathcal{P}_2(M)$ with $\mu_0 \ll m$. Then there exists a unique geodesic $(\mu_t)_{0 \le t \le 1}$ connecting them, given as

$$\mu_t := (F_t)_* \mu_0,$$

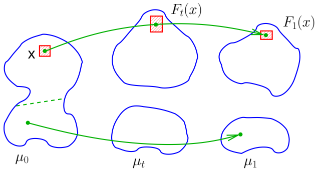
where

$$F_t(x) = \exp_x(t \nabla \varphi(x))$$

with suitable $d^2/2$-convex $\varphi : M \to \mathbb{R}$.

In the case $M = \mathbb{R}^n$ this states that there exists a convex function φ_1 such that

$$F_t(x) = x + t\nabla\varphi(x) = (1-t)x + t\nabla\varphi_1(x).$$



Tangent space:

$T_{\mu_0}\mathcal{P}_2 \quad =$ closure of $\quad \{\Phi = \nabla\varphi : \ M \to TM, \quad \int_M |\nabla\varphi|^2 d\mu_0 < \infty\}$

Gradient and Gradient Flow of the Entropy

Consider $S : \mathcal{P}_2(M) \to \overline{\mathbb{R}}$ with $S(\mu) = \int_M \rho \log \rho \, dm$ if $\mu = \rho \, m$.

The gradient $\nabla S(\mu) \in T_\mu \mathcal{P}_2(M)$ of S at $\mu = \rho \, m$ is given by

$$\nabla S(\mu) = \nabla \log \rho.$$

The gradient flow $\frac{\partial \mu}{\partial t} = -\nabla \, S(\mu)$ on $\mathcal{P}_2(M)$ for S is given by $\mu_t(dx) = \rho_t(x) m(dx)$ where ρ solves the heat equation

$$\frac{\partial}{\partial t} \rho = \triangle \rho \qquad \text{on } M.$$

$\mathbb{R}^n$: Jordan/Kinderlehrer/Otto '98, Otto '01
Riemann (M, g): Ohta '09, Savare '09, Villani '09, Erbar '09
Finsler (M, F, m): Ohta/Sturm '09
Wiener space: Fang/Shao/Sturm '09
Heisenberg group: Juillet '09
Alexandrov spaces: Gigli/Kuwada/Ohta '10
Metric meas. spaces: Ambrosio/Gigli/Savare '11
Discrete spaces: Maas '11

Hessian of the Entropy and Ricci Curvature

M complete Riemannian manifold, m Riemannian volume measure

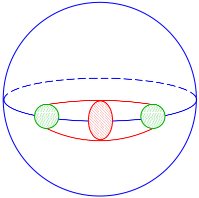
Theorem. (Otto '01, Otto/Villani '00, Cordero/McCann/Schmuckenschläger '01, vRenesse/Sturm '05)

Let $\mathrm{Ent}(\mu) = \int \rho \cdot \log \rho \; dm$ with $\rho = \frac{d\mu}{dm}$. Then

$$\mathrm{Hess\,Ent} \geq K \qquad \Leftrightarrow \qquad \mathrm{Ric}_M \geq K$$

The **proof** depends on the following estimate for the logarithmic determinant $y_t := \log \det dF_t$ of the Jacobian of the transport map:

$$\ddot{y}_t(x) \leq -\frac{1}{n} (\dot{y})_t^2(x) - \mathrm{Ric}(\dot{F}_t(x), \dot{F}_t(x))$$



This inequality is sharp. It describes the effect of curvature on optimal transportation.

Ricci Bounds on Metric Measure Spaces

(X, d) complete separable metric space, m locally finite measure

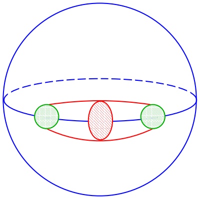
Definition. $CD(K, \infty)$ or $\mathrm{Ric}(X, \mathrm{d}, m) \geq K$

$\iff \quad \forall \mu_0, \mu_1 \in \mathcal{P}_2(X): \ \exists$ geodesic $(\mu_t)_t$ s.t. $\forall t \in [0,1]$:

$$\begin{aligned}\mathrm{Ent}(\mu_t|m) \quad &\leq \quad (1-t)\mathrm{Ent}(\mu_0|m) + t\,\mathrm{Ent}(\mu_1|m) \\ &\quad -\frac{K}{2}\,t(1-t)\,W_2^2(\mu_0,\mu_1)\end{aligned}$$

$$\mathrm{Ent}(\nu|m) = \begin{cases} \int_X \rho \log \rho \, dm & , \text{if } \nu = \rho \cdot m \\ +\infty & , \text{if } \nu \not\ll m \end{cases}$$

$$W_2(\mu_0, \mu_1) = \inf_q \left[\int_{X \times X} d^2(x,y) \, d\,q(x,y) \right]^{1/2}$$



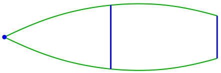
The Curvature-Dimension Condition $CD(0, N)$

Definition. $CD(0, N)$

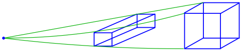
$\Longleftrightarrow \quad \forall \mu_0, \mu_1 \in \mathcal{P}_2(X)$: $\exists$ geodesic $(\mu_t)_t$ s.t. $\forall t \in [0,1]$:

$$S_N(\mu_t|m) \quad \leq \quad (1-t) S_N(\mu_0|m) + t\, S_N(\mu_1|m))$$

Here $S_N(\nu|m) = -\int_X \rho^{1-1/N}\, dm$ for $\nu = \rho \cdot m + \nu_s$



sec $\geq 0 \quad \Longleftrightarrow \quad$ dist concave



ric $\geq 0 \quad \Longleftrightarrow$ vol$^{1/n}$ concave

The Curvature-Dimension Condition $CD^*(K,N)$

Definition. (Bacher/St. JFA 2010) A metric measure space (X,d,m) satisfies the ('reduced') **Curvature-Dimension Condition** $CD^*(K,N)$ for $K,N \in \mathbb{R}$, iff

$\forall\ \rho_0 m,\ \rho_1 m:\ \exists$ geodesic $\rho_t m$ and optimal coupling q satisfying

$$\int_X \rho_t^{1-1/N}(z)\, dm(z) \ \geq\ \int_{X\times X} \Big[\sigma_{K,N}^{(1-t)}(\gamma_0,\gamma_1)\cdot \rho_0^{-1/N}(\gamma_0) + \sigma_{K,N}^{(t)}(\gamma_0,\gamma_1)\cdot \rho_1^{-1/N}(\gamma_1)\Big]\, dq(\gamma_0,\gamma_1)$$

where $\sigma_{K,N}^{(t)}(x,y) = \frac{\sin\left(\sqrt{\frac{K}{N}}\, t\, \mathrm{d}(x,y)\right)}{\sin\left(\sqrt{\frac{K}{N}}\mathrm{d}(x,y)\right)}$. In particular, $\sigma_{0,N}^{(t)}(x,y) = t$.

The Curvature-Dimension Condition $CD^*(K, N)$

Riemannian manifolds:

$$CD^*(K,N) \quad \Longleftrightarrow \quad \mathrm{Ric}_M \geq K \quad \text{and} \quad \dim_M \leq N$$

Weighted Riemannian spaces (M, d, m) with $dm = e^{-V} dvol$:

$$\mathrm{Ric}_M + \mathrm{Hess}\, V - \frac{1}{N-n} DV \otimes DV \;\geq\; K \quad \text{and} \quad \dim_M \leq N$$

Further examples: Ricci limit spaces, Alexandrov spaces, Finsler manifolds (e.g. Banach spaces), Wiener space ($K = 1, N = \infty$).

Constructions: Products, cones, suspensions, warped products.

The Curvature-Dimension Condition $CD^*(K, N)$

Riemannian manifolds:

$$CD^*(K,N) \quad \Longleftrightarrow \quad \mathrm{Ric}_M \geq K \quad \text{and} \quad \dim_M \leq N$$

Weighted Riemannian spaces (M, d, m) with $dm = e^{-V} dvol$:

$$\mathrm{Ric}_M + \mathrm{Hess}\, V - \frac{1}{N-n} DV \otimes DV \;\geq\; K \quad \text{and} \quad \dim_M \leq N$$

Further examples: Ricci limit spaces, Alexandrov spaces, Finsler manifolds (e.g. Banach spaces), Wiener space ($K = 1, N = \infty$).

Constructions: Products, cones, suspensions, warped products.

Theorem (Ketterer 2014) "Cone Theorem"

For any $\kappa \geq 0$ and $N \geq 1$ the following are equivalent

- (X, d, m) satisfies $CD^*(N-1, N)$ and has diameter $\leq \pi$
- The (κ, N)-cone over (X, d, m) satisfies $CD^*(\kappa N, N+1)$

$\kappa = 0$: Euclidean cone; $\kappa = 1$: spherical suspension.

Heat Flow on Metric Measure Spaces

Heat equation on X

- either as gradient flow on $L^2(X, m)$ for the **energy**

$$\mathcal{E}(u) = \frac{1}{2}\int_X |\nabla u|^2 \, dm = \liminf_{v \to u \text{ in } L^2} \frac{1}{2}\int_X (\mathrm{lip}_x v)^2 \, dm(x)$$

with $|\nabla u|$ = minimal weak upper gradient

- or as gradient flow on $\mathcal{P}_2(X)$ for the **relative entropy**

$$\mathrm{Ent}(u) = \int_X u \log u \, dm.$$

Theorem (Ambrosio/Gigli/Savare '11+).

For arbitrary metric measure spaces (X, d, m) satisfying $CD(K, \infty)$ both approaches coincide.

$\mathbb{R}^n$: **Jordan/Kinderlehrer/Otto**
Riemann (M, g): **Ohta, Savare, Villani, Erbar**
Finsler (M, F, m): **Ohta/Sturm**
Alexandrov spaces: **Gigli/Kuwada/Ohta**

Wiener space: **Fang/Shao/Sturm**
Heisenberg group: **Juillet**
Discrete spaces: **Maas, Mielke**

Metric Measure Spaces with Linear Heat Flow

Ricci Bound $\mathrm{CD}(K,\infty)$	$\mathrm{Hess}\,\mathrm{Ent}(.\|m) \geq K$
$\Updownarrow$	
W_2-contraction	$W_2(P_t\mu, P_t\nu) \leq e^{-Kt}\, W_2(\mu,\nu)$
$\Updownarrow$	
Bakry-Émery gradient estimate	$\|\nabla P_t u\|^2 \leq e^{-2Kt}\, P_t\, \|\nabla u\|^2$
$\Updownarrow$	
Bochner Inequality (without N)	$\frac{1}{2}\Delta\|\nabla u\|^2 - \langle \nabla u, \nabla\Delta u\rangle \geq K\cdot\|\nabla u\|^2$

[Ambrosio, Gigli, Savare]

Metric Measure Spaces with Linear Heat Flow

Ricci Bound $CD(K,\infty)$	$\text{Hess}\,\text{Ent}(.\vert m) \geq K$
$\Updownarrow$	
W_2-contraction	$W_2(P_t\mu, P_t\nu) \leq e^{-Kt}\, W_2(\mu,\nu)$
$\Updownarrow$	
Bakry-Émery gradient estimate	$\vert\nabla P_t u\vert^2 \leq e^{-2Kt}\, P_t\, \vert\nabla u\vert^2$
$\Updownarrow$	
Bochner Inequality (without N)	$\frac{1}{2}\Delta\vert\nabla u\vert^2 - \langle \nabla u, \nabla\Delta u\rangle \geq K \cdot \vert\nabla u\vert^2$

- What are the corresponding assertions for finite N?
- Can we deduce a $CD^*(K', N')$-condition for the transformed space (X, d', m) if (X, d, m) satisfies $CD^*(K, N)$ and

$$d'(x,y) = \inf\left\{ \int_0^1 |\dot\gamma_t| \cdot e^{W(\gamma_t)}\, dt : \ \gamma : [0,1] \to X \text{ rect. } , \gamma_0 = x, \gamma_1 = y \right\}$$

(briefly: $d' \approx e^W d$ on infinitesimal scales)

Metric Measure Spaces with Linear Heat Flow

Ricci Bound CD(K,∞)	$\mathrm{Hess\,Ent}(.\vert m) \geq K$
$\Updownarrow$	
W_2-contraction	$W_2(P_t\mu, P_t\nu) \leq e^{-Kt}\, W_2(\mu,\nu)$
$\Updownarrow$	
Bakry-Émery gradient estimate	$\vert\nabla P_t u\vert^2 \leq e^{-2Kt}\, P_t\, \vert\nabla u\vert^2$
$\Updownarrow$	
Bochner Inequality (without N)	$\frac{1}{2}\Delta\vert\nabla u\vert^2 - \langle\nabla u, \nabla\Delta u\rangle \geq K\cdot\vert\nabla u\vert^2$

- What are the corresponding assertions for finite N?
- Can we deduce a $\mathrm{CD}^*(K',N')$-condition for the transformed space (X,d',m) if (X,d,m) satisfies $\mathrm{CD}^*(K,N)$ and

$$d'(x,y) = \inf\left\{\int_0^1 |\dot\gamma_t|\cdot e^{W(\gamma_t)}\,dt:\ \gamma:[0,1]\to X \text{ rect. }, \gamma_0=x, \gamma_1=y\right\}$$

(briefly: $d' \approx e^W d$ on infinitesimal scales)

Well knwon: $\mathrm{CD}^*(K,N)$ for (X,d,m) implies $\mathrm{CD}^*(K',N')$ for the weighted space $(X,d,e^V m)$ with $K' = K - \sup\left[\mathrm{Hess}\,V + \frac{1}{N'-N}\nabla V\otimes\nabla V\right]$

Alternative Introduction

Γ-Calculus of Bakry, Emery, Ledoux

Setting

L linear operator defined on algebra $\mathcal{A}$ of functions on X

e.g. $L = \Delta$ Laplace-Beltrami, $\mathcal{A} = C_c^\infty(M)$, $X =$ Riem.mfd. M

Derived quantities

- Square field operator $\Gamma(f,g) = \frac{1}{2}[L(fg) - fLg - gLf]$
- Hessian $H_f(g,h) = \frac{1}{2}\Big[\Gamma\big(g,\Gamma(f,h)\big) + \Gamma\big(h,\Gamma(f,g)\big) - \Gamma\big(f,\Gamma(g,h)\big)\Big]$
- Γ_2-operator $\Gamma_2(f,g) = \frac{1}{2}[L\Gamma(f,g) - \Gamma(f,Lg) - \Gamma(g,Lf)]$

e.g. $\Gamma(f,g) = \nabla f\, \nabla g, \quad H_f(g,h) = \mathrm{Hess} f(\nabla g, \nabla h),$

$$\Gamma_2(f,f) = \frac{1}{2}\Delta(|\nabla f|^2) - \nabla f\, \nabla \Delta f = \mathrm{Ric}(\nabla f, \nabla f) + \|\nabla^2 f\|_{HS}^2$$

Γ-Calculus of Bakry, Emery, Ledoux

Setting

L linear operator defined on algebra $\mathcal{A}$ of functions on X

e.g. $L = \Delta$ Laplace-Beltrami, $\mathcal{A} = C_c^\infty(M)$, $X =$ Riem.mfd. M

Derived quantities

- Square field operator $\Gamma(f,g) = \frac{1}{2}[L(fg) - fLg - gLf]$
- Hessian $H_f(g,h) = \frac{1}{2}\big[\Gamma\big(g,\Gamma(f,h)\big) + \Gamma\big(h,\Gamma(f,g)\big) - \Gamma\big(f,\Gamma(g,h)\big)\big]$
- Γ_2-operator $\Gamma_2(f,g) = \frac{1}{2}[L\Gamma(f,g) - \Gamma(f,Lg) - \Gamma(g,Lf)]$

e.g. $\Gamma(f,g) = \nabla f \, \nabla g$, $\quad H_f(g,h) = \mathrm{Hess} f(\nabla g, \nabla h)$,

$$\Gamma_2(f,f) = \frac{1}{2}\Delta(|\nabla f|^2) - \nabla f \, \nabla \Delta f = \mathrm{Ric}(\nabla f, \nabla f) + \|\nabla^2 f\|_{HS}^2$$

Ricci tensor

- $R(f)(x) = \inf\left\{\Gamma_2(\tilde{f})(x) : \ \tilde{f} \in \mathcal{A}, \ \Gamma(\tilde{f} - f)(x) = 0\right\}$

Γ-Calculus – Ricci Bounds

Ricci and N-Ricci tensor

- $R(f)(x) = \inf\left\{\Gamma_2(\tilde{f})(x) : \ \tilde{f} \in \mathcal{A}, \ \Gamma(\tilde{f} - f)(x) = 0\right\}$
- $R_N(f)(x) = \inf\left\{\Gamma_2(\tilde{f})(x) - \frac{1}{N}(L\tilde{f})^2(x) : \ \tilde{f} \in \mathcal{A}, \ \Gamma(\tilde{f} - f)(x) = 0\right\}$

Bakry-Emery Condition (Bochner Inequality)

$$
\begin{aligned}
BE(K,N) \quad &:\Leftrightarrow \quad \Gamma_2(f) \geq K \cdot \Gamma(f) + \frac{1}{N}(Lf)^2 \\
&\Leftrightarrow \quad R_N(f) \geq K \cdot \Gamma(f)
\end{aligned}
$$

Γ-Calculus – Transformations

Drift transformation $L'u := Lu + \Gamma(V, u)$

- $R'(u) = R(u) - H_V(u, u)$
- $R'_{N'}(u) \geq R_N - H_V(u, u) - \frac{1}{N' - N} \Gamma(V, u)^2$

Cor. BE(K,N) for L $\Rightarrow$ BE(K',N') for L'

Ex. L generator of Dirichlet form $\int \Gamma(u) dm$ on $L^2(X, m)$
$\Rightarrow$ L' generator of Dirichlet form $\int \Gamma(u) e^V dm$ on $L^2(X, e^V m)$

Aim. Independent choice of weights

$$\mathcal{E}'(u) = \int \Gamma(u) e^{V - 2W} dm \quad \text{on} \quad \mathcal{H}' = L^2(X, e^V m)$$

e.g.

- $W = 0$: drift transformation
- $V = 2W$: time change
- $V = NW$: conformal transformation

Γ-Calculus – Transformations

Assume that $\mathcal{E}(u) = \int \Gamma(u) dm$ on $\mathcal{H} = L^2(X, m)$ satisfies $BE(K, N)$.

Theorem (St. 2014)

Then

$$\mathcal{E}'(u) = \int \Gamma(u) e^{V-2W} dm \quad \text{on} \quad \mathcal{H}' = L^2(X, e^V m)$$

satisfies $BE(K', N')$.

- $W = 0$ (Drift Transformation)

$$K' = K - \sup \left[\mathrm{Hess} V(\nabla f, \nabla f) + \frac{1}{N' - N} \langle \nabla V, \nabla f \rangle^2 \right] / |\nabla f|^2$$

- $V = 2W$ (Time Change)

$$K' = \inf \left[e^{-2W} K + \frac{1}{2} \Delta e^{-2W} - N^* |\nabla e^{-W}|^2 \right]$$

- $V = NW$ (Conformal Transformation)

$$N' = N, \; K' = \ldots$$

$CD^*(K,N)$ and $BE(K,N)$ with Finite N

What do we gain if we have $BE(K,N)$ instead of $BE(K,\infty)$?

Consider evolution

$$dX_t = \sqrt{2\alpha}dB_t - \nabla V(X_t)\,dt$$

for $A = \alpha\Delta - \nabla V \cdot \nabla$ on n-dimensional Riem (M,g). Then

$$BE(K,\infty) \quad \Longleftrightarrow \quad \alpha\mathrm{Ric} + \mathrm{Hess}\,V \geq K$$

$$BE(K,N) \quad \Longleftrightarrow \quad \alpha\mathrm{Ric} + \mathrm{Hess}\,V - \frac{1}{N-\alpha n}(\nabla V \otimes \nabla V) \geq K$$

For $\alpha \to 0$: $\quad dX_t = -\nabla V(X_t)dt$

$$\mathrm{Hess}\,V - \frac{1}{N}(\nabla V \otimes \nabla V) \geq K$$

(K,N)-Convexity

Def V is (K,N)-convex

$\iff$ Hess $V - \frac{1}{N}(\nabla V \otimes \nabla V) \geq K$

$\iff$ Hess $U_N \leq -\frac{K}{N} \cdot U_N$ where $U_N(x) := \exp\left(-\frac{1}{N} V(x)\right)$

$\iff$ $U_N(\gamma_t) \geq \sigma_{K,N}^{(1-t)}(|\dot\gamma|) \cdot U_N(\gamma_0) + \sigma_{K,N}^{(t)}(|\dot\gamma|) \cdot U_N(\gamma_1)$

Example. For $N > 0$ and $K > 0$

$$V(x) \; = \; -N \log\cos\left(x\sqrt{K/N}\right) \quad \geq \; \frac{K}{2}x^2$$

defined on $(-\sqrt{N/K}\pi/2, \sqrt{N/K}\pi/2)$

(K, N)-Convexity

Thm. A smooth curve $x : [0, \infty) \to M$ is a solution to the gradient flow equation

$$\dot{x}_t = -\nabla V(x_t)$$

if and only if the Evolution Variation Inequality $EVI_{K,N}$ holds:

$$\frac{d}{dt} s^2_{K/N}\left(\frac{1}{2} d(x_t, z)\right) + K \cdot s^2_{K/N}\left(\frac{1}{2} d(x_t, z)\right) \le \frac{N}{2}\left(1 - \frac{U_N(z)}{U_N(x_t)}\right)$$

where $s_K(r) = \sin(\sqrt{K} r)/\sqrt{K}$, $c_K(r) = \cos(\sqrt{K} r)$

(K, N)-Convexity

Corollary. Let $(x_t), (y_t)$ be two gradient flows of V starting from x_0 resp. y_0. Then for all $s, t \geq 0$:

$$\begin{aligned} s_{K/N}^2\left(\frac{1}{2}d(x_t, y_s)\right) \quad &\leq \quad e^{-K(s+t)} \cdot s_{K/N}^2\left(\frac{1}{2}d(x_0, y_0)\right) \\ &\quad + \frac{N}{K}\left(1 - e^{-K(s+t)}\right)\frac{\left(\sqrt{t} - \sqrt{s}\right)^2}{2(s+t)} \end{aligned}$$

If $K = 0$: $\qquad d^2(x_t, y_s) \leq d^2(x_0, y_0) + 4N\left(\sqrt{t} - \sqrt{s}\right)^2$

Everything makes sense also on metric spaces!
E.g. for $V = Ent(.)$ on $M = \mathcal{P}_2(X, d)$.

Entropic Curvature-Dimension Condition

Def. A metric measure space (X, d, m) satisfies $CD^e(K, N)$

$\Longleftrightarrow$ Ent(.) is (K,N)-convex on $\mathcal{P}_2(X, d)$

$\Longleftrightarrow$ $\forall \mu_0, \mu_1 \in \mathcal{P}_2(X, d)$: $\exists$ connecting geodesic $(\mu_t)_t$ s.t.
$\forall t \in [0, 1]$:

$$U_N(\mu_t) \;\geq\; \sigma_{K,N}^{(1-t)}(|\dot{\mu}|) \cdot U_N(\mu_0) + \sigma_{K,N}^{(t)}(|\dot{\mu}|) \cdot U_N(\mu_1)$$

where

$$U_N(\mu) \;=\; \exp\left(-\frac{1}{N}\mathrm{Ent}(\mu)\right)$$

Thm. For non-branching mms

$$CD^e(K, N) \qquad \Longleftrightarrow \qquad CD^*(K, N)$$

Entropic Curvature-Dimension Condition

Thm. The following are equivalent

- strong $CD^*(K,N)$-space and linear heat flow
- strong $CD^e(K,N)$-space and linear heat flow
- X is length space and each $\mu \in \mathcal{P}_2$ is starting point of $\mathrm{EVI}_{K,N}$-gradient flow for $\mathrm{Ent}(.)$

Def. $\mathrm{RCD}^*(K,N) \;=\; \mathrm{CD}^*(K,N)$ and linear heat flow

Thm.

(i) $\mathrm{RCD}^*(K,N)$ is preserved under convergence of mms

(ii) $\mathrm{RCD}^*(K,N)$ is preserved under tensorization of mms

(iii) $\mathrm{RCD}^*(K,N)$ holds globally if it holds locally

RCD*(K, N)-Spaces

Theorem (Erbar/Kuwada/St. 2013) "W_2-contraction"

$$s^2_{K/N}\left(\frac{1}{2}W_2(P_t\mu,P_s\nu)\right) \le e^{-K(s+t)}\cdot s^2_{K/N}\left(\frac{1}{2}W_2(\mu,\nu)\right) + \frac{N}{K}\left(1-e^{-K(s+t)}\right)\frac{\left(\sqrt{t}-\sqrt{s}\right)^2}{2(s+t)}.$$

with $s_K(r)=\sin(\sqrt{K}r)/\sqrt{K}$

Corollary: With $\tau(s,t)=2(t+\sqrt{ts}+s)/3$

$$W_2(P_t\mu,P_s\nu)^2 \le e^{-K\tau(s,t)}W_2(\mu,\nu)^2+2N\frac{1-e^{-K\tau(s,t)}}{K\tau(s,t)}\left(\sqrt{t}-\sqrt{s}\right)^2,$$

Theorem (Erbar/Kuwada/St. 2013) "Bakry-Ledoux gradient estimate"

$$|\nabla P_t f|^2+\frac{4Kt^2}{N\left(e^{2Kt}-1\right)}|\Delta P_t f|^2 \le e^{-2Kt}P_t\left(|\nabla f|^2\right)$$

$\mathrm{RCD}^*(K, N)$-Spaces

Theorem (Erbar/Kuwada/St. 2013) "Bochner Inequality"

$$\frac{1}{2}\Delta|\nabla u|^2 - \langle \nabla u, \nabla \Delta u\rangle \geq K \cdot |\nabla u|^2 + \frac{1}{N} \cdot |\Delta u|^2$$

- **Finsler spaces: Ohta-Sturm (2012+)**
- **Alexandrov spaces: Gigli-Kuwada-Ohta (2010+), Zhang-Zhu (2012)**
- $CD^e(K, \infty)$**-spaces: Ambrosio-Gigli-Savaré (2012+)**
- $CD^e(K, N)$**-spaces: Ambrosio-Mondino-Savaré (work in progress)**

Corollary (Garofalo/Mondino 2013)

$$\Delta(\log P_t f) \geq -\frac{N}{2t}$$

Li-Yau gradient estimate, differential Harnack inequality, Gaussian heat kernel estimates

RCD$^*(K, N)$-Spaces

Thm. For length space with linear heat flow the following are equivalent

(i) Strong $CD^*(K, N)$

(ii) Strong $CD^e(K, N)$

(iii) Existence of an $EVI_{K,N}$-gradient flow for the entropy starting from each point $\mu \in \mathcal{P}_2$

(iv) W_2-contraction estimate with parameters K, N

(v) Bakry-Ledoux gradient estimate

(vi) Bochner inequality BE(K, N)

Consequences of (K,N)-Convexity

Thm. (N-LogSob Inequality) Assume that S is (K,N)-convex for some $0 < K, N < \infty$ and that $\inf S = 0$. Then

$$(\nabla S)^2 \geq KN \cdot \left[\exp\left(\frac{2}{N}S\right) - 1\right] \quad \geq 2KS$$

Thm. (N-HWI Inequality) Assume that S is (K,N)-convex for some $0 < K, N < \infty$ and that $\inf S = S(\bar{x}) = 0$. Then $\forall x_0$

$$|\nabla S|(x_0) \cdot d(x_0, \bar{x}) - \frac{K}{2} d^2(x_0, \bar{x}) \geq N \cdot \left[\exp\left(\frac{1}{N}S(x_0)\right) - 1\right] \quad \geq S$$

Thm. (N-Talagrand Inequality) Assume that S is (K,N)-convex for some $0 < K, N < \infty$ and that $\inf S = S(\bar{x}) = 0$. Then $\forall x_0$

$$S(x_0) \geq -N \log\cos\left(\sqrt{\frac{K}{N}}\, d(x_0, \bar{x})\right) \quad \geq \frac{K}{2} d^2(x_0, \bar{x})$$

Transformation of $RCD^*(K,N)$-Spaces

Given mms (X,d,m), functions V,W on X. Consider mms (X,d',m') with $m' = e^V m$ and

$$d'(x,y) = \inf\left\{ \int_0^1 |\dot\gamma_t| \cdot e^{W(\gamma_t)}\, dt : \ \gamma : [0,1] \to X \text{ rectifiable}, \gamma_0 = x, \gamma_1 = y \right\}.$$

Associated Dirichlet form: $\int |\nabla u|^2 e^{V-2W}\, dm$ on $L^2(X, e^V m)$

Theorem (St. 2014)

If (X,d,m) satisfies $RCD^*(K,N)$ then for each $N' > N$ there exists K' s.t. (X,d',m') satisfies $RCD^*(K',N')$.

- $W = 0$ (Drift Transformation)

$$K' = K - \sup\left[\mathrm{Hess}\, V(\nabla f, \nabla f) + \frac{1}{N'-N}\langle \nabla V, \nabla f\rangle^2\right] / |\nabla f|^2$$

- $V = 2W$ (Time Change)

$$K' = \inf\left[e^{-2W} K + \frac{1}{2}\Delta e^{-2W} - N^* |\nabla e^{-W}|^2\right]$$

- $V = NW$ (Conformal Transformation)

$$N' = N, \ K' = \dots$$

Geometry/Analysis on $RCD^*(K,N)$-Spaces

Bishop-Gromov Volume Growth Estimate $\quad s(r) = \frac{\partial}{\partial r} m(B_r(x_0))$

$$s(r)/s(R) \quad \geq \quad \sin\left(\sqrt{\frac{K}{N-1}}\,r\right)^{N-1} \Big/ \sin\left(\sqrt{\frac{K}{N-1}}\,R\right)^{N-1}$$

Bonnet-Myers Diameter Bound

$$\mathrm{diam}(X) \quad \leq \quad \sqrt{\frac{N-1}{K}} \cdot \pi$$

Poincaré / Lichnerowicz Inequality

$$\lambda_1 \geq \frac{N}{N-1} K$$

Geometry of $RCD^*(K, N)$-Spaces

Theorem (Gigli 2013) "Splitting Theorem"

If (X, d, m) satisfies $RCD^*(0, N)$ and contains a line then

$$X = \mathbb{R} \times X'$$

for some $RCD^*(0, N-1)$-space (X', d', m').

Theorem (Ketterer 2014) "Maximal Diameter Theorem"

If (X, d, m) satisfies $RCD^*(N-1, N)$ and has diameter π then X is the spherical suspension of some $RCD^*(N-2, N-1)$-space (X', d', m').

Theorem (Mondino/Naber 2014)

If (X, d, m) satisfies $RCD^*(K, N)$ then $\exists$ integer $n \leq N$ s.t. for m-a.e. $x \in X$ the tangent cone at x is unique and isometric to $\mathbb{R}^n$.

The End