Robust Arbitrage under Uncertainty in Discrete Time

Marco Frittelli

Joint with **Matteo Burzoni** and **Marco Maggis**

Workshop on Mathematical Finance - Banff, May, 2014

Outline

- Motivation
- Feasible Markets
- Robust Arbitrage
- Arbitrage de la classe $\mathcal{C}$ and Open Arbitrage
- Defragmentation
- Robust Arbitrage and Martingale Measures
- Filtration Enlargement

Pricing

We have two extreme cases.

A We are completely sure about the reference probability measure P. In this case, the classical notion of No Arbitrage or NFLVR can be successfully applied.

B We face complete uncertainty about any probabilistic model and therefore we describe our model independently by any probability. In this case we might use a model independent notion of No Arbitrage [Acciaio, Beiglböck, Penkner, Schachermayer 2013].

Pricing under uncertainty

Between these two cases, there is the possibility to accept that the model could be described in a probabilistic setting, but we cannot assume the knowledge of a specific reference probability measure (Knightian uncertainty).

Peng, Denis Martini, Touzi Zhang, Soner, Hobson, Dolinsky Soner, Riedel, Bouchard Nutz....

In this framework, we have several approaches and we may pose several questions:

Pricing w.r.to a given set of Probabilities

- Which is a good notion of a common arbitrage opportunity ?

That is, an arbitrage opportunity for all **admissible** probabilistic models (one single H that works as an arbitrage for all admissible models).

The answer is provided (in discrete time) by Bouchard-Nutz '13.

To pose this question we need to know **a priori** which are the admissible models, i.e. we have to exhibit a subset $\mathcal{R} \subseteq \mathcal{P}$ of probabilities.

Pricing w.r.to a given set R of Probabilities [BN13]

Definition

- $A \in \Omega$ is $\mathcal{R}$-*polar* set if $A \subseteq A'$, for some $A' \in \mathcal{F}$ such that $P(A') = 0 \quad \forall P \in \mathcal{R}$.
- A property holds $\mathcal{R}$-**quasi surely** if it holds outside any $\mathcal{R}$-polar set.
- A strategy $H \in \mathcal{H}$ is an **Arbitrage w.r.to** $\mathcal{R}$ if
  - $V_T(H) \geq 0$ $\mathcal{R}$-q.s.;
  - there exists a $P \in \mathcal{R}$ such that $P(V_T(H) > 0) > 0$.

Theorem (I FTAP)

No Arbitrage w.r.to $\mathcal{R}$ $\quad\Leftrightarrow\quad$ $\mathcal{R}$ *and* $\mathcal{M}$ *have the same polar sets.*

Our approach: pricing with respect to most probabilities

1. Which are the markets that are feasible (in the sense that the properties of the market are nice for "**most**" probabilistic models) ?
   - Given a market (described without reference to a probability), the induced set of probabilities for which No Arbitrage holds will determine if the market itself is feasible or not.
2. Once feasibility is established, can we determine a **robust** notion of No Arbitrage, i.e. one strategy H regarded as an Arbitrage in most probabilistic models ?

Our approach: pricing with respect to most probabilities

- To answer these two questions we do not need to exhibit a priori a subset of probabilities.
- On the contrary, given a market (described without reference to a probability), the induced set of No Arbitrage models (probabilities) - for that market - will determine if the market itself is feasible or not.
- What is needed here is a good notion of "**most**" probabilistic models.
- But the set of probabilities that works are **intrinsic** to the model.

Underlying Model

Let Ω be a Polish space and $\mathcal{F} = \mathcal{B}(\Omega)$ the Borel sigma algebra.
We fix

- a finite time horizon $T \geq 1$,
- a finite set of time indices $I = \{0, ..., T\}$,
- a filtration $\{\mathcal{F}_t\}_{t \in I}$ with $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_T \subseteq \mathcal{F}$

The market consists of a non-risky asset:

$$S_t^0 = 1 \text{ for all } t \in I,$$

and $d \geq 1$ risky assets

$$S^j = (S_t^j)_{t \in I}, \quad j = 1, ..., d,$$

that are real-valued adapted stochastic processes.

The value process

We will be considering self financing strategies with **zero initial value**, and therefore we may assume that a trading strategy $H = (H_t)_{t \in I}$ is a $\mathbb{R}^d$-valued predictable stochastic process: $H_t = [H_t^1, ..., H_t^d]$ and the value process $V(H) = (V_t(H))_{t \in I}$ is given by:

$$V_t(H) = (H \circ S)_t = \sum_{u=1}^{t} H_u(S_u - S_{u-1}),\ t = 1, ..., T$$

We indicate with $\mathcal{H}$ the class of all trading strategies.

Notations

1. $\mathcal{P} := \mathcal{P}(\Omega)$ is the set of all probabilities on $(\Omega, \mathcal{F})$ and $C_b := C_b(\Omega)$ the space of continuous and bounded functions on Ω.
2. $\sigma(\mathcal{P}, C_b)$ is the weak* topology on $\mathcal{P}$.
3. The *support* of $P \in \mathcal{P}$ is

$$supp(P) = \bigcap \{C \mid C \text{ closed and } P(C) = 1\}$$

and P **has full support** if

$$supp(P) = \Omega.$$

Notations

$$
\begin{aligned}
\mathcal{P}_e(P) &:= \{Q \in \mathcal{P} \mid Q \sim P\} \\
\mathcal{M} &:= \{Q \in \mathcal{P} \mid S \text{ is a } Q\text{-martingale}\} \\
\mathcal{M}_e(P) &:= \{Q \in \mathcal{M} \mid Q \sim P\} \\
\mathcal{P}_+ &:= \{Q \in \mathcal{P} \mid Q \text{ has full support}\} \\
\mathcal{M}_+ &:= \mathcal{M} \cap \mathcal{P}_+,
\end{aligned}
$$

$\mathcal{M}_+$ is the set of *martingale probability measures with full support.*

Notation: No R-Arbitrage

We extend the classical notion of arbitrage respect to a single probability measure $P \in \mathcal{P}$ ($V_T(H) \geq 0$ P-a.s. and $P(V_T(H) > 0) > 0$) to a class of probabilities $\mathcal{R} \subseteq \mathcal{P}$ as follows:

Definition ($\mathcal{R}$-Arbitrage)

Let $\mathcal{R} \subseteq \mathcal{P}$.The market admits $\mathcal{R}$-Arbitrage if

- for all $P \in \mathcal{R}$ there exists a P-Arbitrage.

$NA(\mathcal{R})$ holds if for some $P \in \mathcal{R}$, $NA(P)$ holds true.

- The presence of an $\mathcal{R}$-Arbitrage only implies that for each P there exists a trading strategy H^P which is a P-Arbitrage.
- The existence of one single trading strategy H that realizes an arbitrage for a "large" class of $P \in \mathcal{P}$ will be analyzed when dealing with a Robust Arbitrage.

Assumption

A minimal assumption on the financial market is $NA(\mathcal{P})$, i.e. the existence of at least one probability $P \in \mathcal{P}$ that does not allow any P-Arbitrage. Set:

$$\mathcal{P}_0 = \{P \in \mathcal{P} \mid \mathcal{M}_e(P) \neq \varnothing\}.$$

It is clear, from the I FTAP [DMW90], that:

$$NA(\mathcal{P}) \Leftrightarrow \mathcal{M} \neq \varnothing \Leftrightarrow \mathcal{P}_0 \neq \varnothing$$

and that these conditions are not equivalent to the absence of a Global Arbitrage (No H s.t. $V_T(H)(\omega) > 0$ for all $\omega \in \Omega$).

Assumption: in the sequel it is always assumed that

$$\mathcal{P}_0 \neq \varnothing$$

Feasible Markets

Examples

$$\mathcal{P}_0 = \{P \in \mathcal{P} \mid NA(P)\} = \{P \in \mathcal{P} \mid \mathcal{M}_e(P) \neq \varnothing\}$$

$$a) \quad 2 \quad \to \quad \begin{cases} 3 & \omega_1 \\ 2 & \omega_2 \\ 1 & \omega_3 \end{cases} ; \quad b) \quad 2 \to \begin{cases} 3 & \omega_1 \\ 2 & \omega_2 \\ 2 & \omega_3 \end{cases} ;$$

$$c) \quad 2 \quad \to \quad \begin{cases} 7 & \omega_1 \\ 3 & \omega_2 \\ 1 & \omega_3 \end{cases} ; \quad d) \quad 2 \to \begin{cases} 7 & \omega_1 \\ 3 & \omega_2 \\ 2 & \omega_3 \end{cases} .$$

a) $\mathcal{P}_0 = \mathcal{P} \diagdown ([\delta_1, \delta_2) \cup (\delta_2, \delta_3])$

b) $\mathcal{P}_0 = [\delta_2, \delta_3]$ Is it feasible ?

c) $\mathcal{P}_0 = \mathcal{P} \diagdown ([\delta_1, \delta_2] \cup \delta_3)$

d) $\mathcal{P}_0 = \delta_3$ Is it feasible ?

$\mathcal{P}$ is the set of all probability measures on $(\Omega, \mathcal{F})$

Feasible Markets

Given a financial market $(\Omega; \mathcal{F}; (\mathcal{F}_t)_t; S; \mathcal{H})$, it is possible:

- That for only very few probability measures - the extreme case being $|\mathcal{P}_0| = 1$) the market is arbitrage free.
- Or that the financial market is very "well posed": for "most" probability measures no arbitrage is assured - the extreme case being $\mathcal{P}_0 = \mathcal{P}$.

Definition

The market is feasible if

$$\overline{\mathcal{P}_0} = \mathcal{P}$$

- This definition depends on the topology on $\mathcal{P}$.

First characterization of Feasibility: weak* topology

If Ω is a Polish space then TFAE:

1. $\mathcal{M} \cap \mathcal{P}_+ \neq \emptyset$
2. $NA(\mathcal{P}_+)$
3. $\mathcal{P}_0 \cap \mathcal{P}_+ \neq \emptyset$
4. $\overline{\mathcal{P}_0 \cap \mathcal{P}_+}^{\sigma(\mathcal{P}, C_b)} = \mathcal{P}$
5. $\overline{\mathcal{P}_0}^{\sigma(\mathcal{P}, C_b)} = \mathcal{P}$

where:
$\mathcal{M}$ is the set of all martingale measures
$\mathcal{P}_+$ is the set of probability measures with full support
$NA(\mathcal{P}_+)$ means: $NA(P)$ for some $P \in \mathcal{P}_+$

First characterization of Feasibility: norm topology

If Ω is countable then TFAE:

1. $\mathcal{M} \cap \mathcal{P}_+ \neq \emptyset$
2. $NA(\mathcal{P}_+)$
3. $\mathcal{P}_0 \cap \mathcal{P}_+ \neq \emptyset$
4. $\overline{\mathcal{P}_0}^{\|\cdot\|} = \mathcal{P}$

where:
the closure is taken in the topology induced by the total variation norm.
$\mathcal{M}$ is the set of all martingale measures
$\mathcal{P}_+$ is the set of probability measures with full support
$NA(\mathcal{P}_+)$ means: $NA(P)$ for some $P \in \mathcal{P}_+$

Instead of looking at the set of probabilities for which the market is free of arbitrage opportunities, we discuss the existence of a common arbitrage opportunity that works for a **large** class of probabilistic models.

Robust Arbitrage

Definition

A market admits a **Robust Arbitrage** if there exists a strategy $H \in \mathcal{H}$ and a non empty τ-open set $\mathcal{U} \subseteq \mathcal{P}$ such that

$$V_T(H) \geq 0 \ P\text{-a.s.} \ \forall P \in \mathcal{U} \quad \text{and} \quad P(V_T(H) > 0) > 0 \ \forall P \in \mathcal{U}.$$

- No a priori fixed class of probability measures

Robust Arbitrage

Definition

A market admits a **Robust Arbitrage** if there exists a strategy $H \in \mathcal{H}$ and a non empty τ-open set $\mathcal{U} \subseteq \mathcal{P}$ such that

$$V_T(H) \geq 0 \ P\text{-a.s.} \ \forall P \in \mathcal{U} \quad \text{and} \quad P(V_T(H) > 0) > 0 \ \forall P \in \mathcal{U}.$$

- No a priori fixed class of probability measures
- This definition depends on the topology τ on $\mathcal{P}$.
- If $\tau = \sigma(\mathcal{P}, C_b)$ we refer to Robust Arbitrage
- If τ is induced by $\|\cdot\|$ we refer to Strong Robust Arbitrage.

Robust Arbitrage

Definition

A market admits a **Robust Arbitrage** if there exists a strategy $H \in \mathcal{H}$ and a non empty τ-open set $\mathcal{U} \subseteq \mathcal{P}$ such that

$$V_T(H) \geq 0 \ P\text{-a.s.} \ \forall P \in \mathcal{U} \quad \text{and} \quad P(V_T(H) > 0) > 0 \ \forall P \in \mathcal{U}.$$

- No a priori fixed class of probability measures
- This definition depends on the topology τ on $\mathcal{P}$.
- If $\tau = \sigma(\mathcal{P}, C_b)$ we refer to Robust Arbitrage
- If τ is induced by $\|\cdot\|$ we refer to Strong Robust Arbitrage.
- If $(H, \mathcal{U})$ is a Robust Arbitrage and we disregard any finite subset of probabilities then H remains a Robust Arbitrage.
- If $(H, \mathcal{U})$ is a Robust Arbitrage, $\mathcal{U}$ will contain a full support probability P under which H is a P-Arbitrage.
- We now show an equivalent formulation easier to interpret.

Arbitrage de la classe C and Open Arbitrage

Arbitrage de la classe C

Set: $\mathcal{V}_H^+ := \{\omega \in \Omega \mid \; V_T(H) > 0\}$ and recall that $V_0(H) = 0$.

Definition

Let $\mathcal{C}$ be a class of subsets of Ω with $\varnothing \notin \mathcal{C}$. $H \in \mathcal{H}$ is an Arbitrage de la classe $\mathcal{C}$ if $V_T(H)(\omega) \geq 0 \; \forall \omega \in \Omega$ and $\mathcal{V}_H^+$ **contains a set** de la classe $\mathcal{C}$.

- The class $\mathcal{C}$ has the role to translate mathematically the meaning of a "true gain".

Arbitrage de la classe C

Set: $\mathcal{V}_H^+ := \{\omega \in \Omega \mid \ V_T(H) > 0\}$ and recall that $V_0(H) = 0$.

Definition

Let $\mathcal{C}$ be a class of subsets of Ω with $\varnothing \notin \mathcal{C}$. $H \in \mathcal{H}$ is an Arbitrage de la classe $\mathcal{C}$ if $V_T(H)(\omega) \geq 0 \ \forall \omega \in \Omega$ and $\mathcal{V}_H^+$ **contains a set** de la classe $\mathcal{C}$.

- The class $\mathcal{C}$ has the role to translate mathematically the meaning of a "true gain".
- When a "reference Probability" P is given, then a true gain is: $P(V_T(H) > 0) > 0$ and No Arbitrage is:

  no losses $P(V_T(H) < 0) = 0$ implies no true gain $P(V_T(H) > 0) = 0$.
- When a subset $\mathcal{R}$ of probability measures is given, one may replace the P-a.s. conditions above with $\mathcal{R}$-q.s conditions, as in [BN13].
- However, if we do not want to (or can not) rely on a priory assigned set of probability measures, then we may use a class $\mathcal{C}$ to represent true gains, but then we require $V_T(H)(\omega) \geq 0 \ \forall \omega \in \Omega$.

Examples of Arbitrage de la Classe C

- H is a Pointwise Arbitrage when $\mathcal{C} = \{C \in \mathcal{F} \mid C \neq \varnothing\}$
- H is an **Open Arbitrage** if $\mathcal{C} = \{C \in \mathcal{B}(\Omega) \mid C \text{ open non-empty}\}$
- H is a Global Arbitrage when $\mathcal{C} = \{\Omega\}$

Examples of Arbitrage de la Classe C

- H is a Pointwise Arbitrage when $\mathcal{C} = \{C \in \mathcal{F} \mid C \neq \varnothing\}$
- H is an **Open Arbitrage** if $\mathcal{C} = \{C \in \mathcal{B}(\Omega) \mid C$ open non-empty$\}$
- H is a Global Arbitrage when $\mathcal{C} = \{\Omega\}$
- H is a $\mathcal{R}$-q.s. Arbitrage when $\mathcal{C} = \{C \in \mathcal{F} \mid P(C) > 0$ for some $P \in \mathcal{R}\}$, for a fixed family $\mathcal{R} \subseteq \mathcal{P}$. Then No $\mathcal{R}$-q.s. Arbitrage means:

$$H \in \mathcal{H} \text{ such that } V_T(H) \geq 0 \ \forall \omega \in \Omega \Rightarrow V_T(H) = 0 \ \mathcal{R}\text{-q.s.}$$

Examples of Arbitrage de la Classe C

- H is a Pointwise Arbitrage when $\mathcal{C} = \{C \in \mathcal{F} \mid C \neq \varnothing\}$
- H is an **Open Arbitrage** if $\mathcal{C} = \{C \in \mathcal{B}(\Omega) \mid C$ open non-empty$\}$
- H is a Global Arbitrage when $\mathcal{C} = \{\Omega\}$
- H is a $\mathcal{R}$-q.s. Arbitrage when $\mathcal{C} = \{C \in \mathcal{F} \mid P(C) > 0$ for some $P \in \mathcal{R}\}$, for a fixed family $\mathcal{R} \subseteq \mathcal{P}$. Then No $\mathcal{R}$-q.s. Arbitrage means:

$$H \in \mathcal{H} \text{ such that } V_T(H) \geq 0 \ \forall \omega \in \Omega \Rightarrow V_T(H) = 0 \ \mathcal{R}\text{-q.s.}$$

- H is a P-a.s. Arbitrage if $\mathcal{C} = \{C \in \mathcal{F} \mid P(C) > 0\}$ for fixed $P \in \mathcal{P}$. Then No P-a.s. Arbitrage means:

$$H \in \mathcal{H} \text{ such that } V_T(H) \geq 0 \ \forall \omega \in \Omega \Rightarrow V_T(H) = 0 \ P\text{-a.s.}$$

Obviously,

$$\text{No Pointwise} \Rightarrow \text{No A de la Classe } C \ \Rightarrow \text{No Global}$$

and A de la Classe C are not necessarily related to a probabilistic model.

Open and Robust Arbitrage

Proposition

Let $H \in \mathcal{H}$ be an admissible trading strategy. Then:

$$\begin{array}{lcl} H \text{ is a Strong Robust Arbitrage} & \Longleftrightarrow & H \text{ is a Pointwise Arbitrage} \\ \Uparrow & & \\ H \text{ is a Robust Arbitrage} & \Longleftrightarrow & H \text{ is an Open Arbitrage} \end{array}$$

In addition, if H is a Robust Arbitrage then $V_T(H)(\omega) \geq 0$ for all $\omega \in \Omega$.

Defragmentation

- No Open Arbitrage in one period $[t-1, t]$ is not equivalent to No Open Arbitrage on $[0, T]$.
- Many one period Pointwise Arbitrage (which may not be Open Arbitrage) may create an Open Arbitrage on $[0, T]$.

Defragmentation

$$
\begin{array}{cccccc}
 & & 3 \rightarrow & \cdots & \omega \in A_1 \\
 & \nearrow & & 5 & \omega \in A_2 \\
 & & & \nearrow & & \\
S^1 = 2 & \rightarrow & 2 & & & \\
 & & & \searrow & & \\
 & \searrow & & 1 & \omega \in A_3 \\
 & & 1 \rightarrow & \cdots & \omega \in A_4
\end{array}
\qquad
\begin{array}{cccccc}
 & & 7 \rightarrow & \cdots & \omega \in A_1 \\
 & \nearrow & & 3 & \omega \in A_2 \\
 & & & \nearrow & & \\
S^2 = 2 & \rightarrow & 2 & & & \\
 & & & \searrow & & \\
 & \searrow & & 1 & \omega \in A_3 \\
 & & 1 \rightarrow & \cdots & \omega \in A_4
\end{array}
$$

Consider $H_1 = (-1, +1)$ and $H_2 = (\mathbf{1}_{A_2 \cup A_3}, -\mathbf{1}_{A_2 \cup A_3})$. Then

$$H_1 \cdot (S_1 - S_0) = 4\mathbf{1}_{A_1} \text{ and } H_2 \cdot (S_2 - S_1) = 2\mathbf{1}_{A_2}.$$

Selecting ad hoc the sets A_i, an Open Arbitrage can be obtained only by a two steps strategy, while in each step we have a Pointwise Arbitrage.

Pointwise

Lemma

The strategy $H \in \mathcal{H}$ is a Pointwise Arbitrage if and only if there exists a time $t \in I_1$ and $\alpha : \Omega \to \mathbb{R}^d$, $\mathcal{F}_{t-1}$-measurable, such that

$$\begin{array}{ll} \alpha(\omega) \cdot \Delta S_t(\omega) \geq 0 & \forall\, \omega \in \Omega \\ \alpha(\omega) \cdot \Delta S_t(\omega) > 0 & \text{on } A \in \mathcal{F}_t,\ A \neq \varnothing \end{array}$$

Defragmentation of Open Arbitrage

Lemma (Defragmentation)

The strategy H is an Open Arbitrage if and only if there exists:

- *a finite family $\{U_t\}_{t \in I}$ with $U_t \in \mathcal{F}_t$, $U_t \cap U_s = \varnothing$ for every $t \neq s$ and $\bigcup_{t \in I} U_t$ contains an open set;*
- *a strategy $\widehat{H}$ such that for any $U_t \neq \varnothing$ we have $\widehat{H}_t \cdot (S_t - S_{t-1}) > 0$ on U_t.*

Robust Arbitrage and Martingale Measures

Simple Implications: Arbitrage and Martingale Measures

- No Pointwise Arbitrage $\Longrightarrow \mathcal{M}_+ \neq \emptyset \Longrightarrow$ No Robust Arbitrage (w.r.to $\sigma(\mathcal{P}, C_b)$)

Simple Implications: Arbitrage and Martingale Measures

- No Pointwise Arbitrage $\Longrightarrow \mathcal{M}_+ \neq \emptyset \Longrightarrow$ No Robust Arbitrage (w.r.to $\sigma(\mathcal{P}, C_b)$)
- For Ω countable, No Pointwise Arbitrage $\Longleftrightarrow \mathcal{M}_+ \neq \emptyset$.

Simple Implications: Arbitrage and Martingale Measures

- No Pointwise Arbitrage $\Longrightarrow \mathcal{M}_+ \neq \emptyset \Longrightarrow$ No Robust Arbitrage (w.r.to $\sigma(\mathcal{P}, C_b)$)
- For Ω countable, No Pointwise Arbitrage $\Longleftrightarrow \mathcal{M}_+ \neq \emptyset$.
- Examples show that:
  - $\mathcal{M}_+ \neq \emptyset \not\Rightarrow$ No pointwise arbitrage
  - No Global Arbitrage and No Open Arbitrage does not imply the existence of a martingale measure

$$\text{No Open Arbitrage} \not\Rightarrow \mathcal{M} \neq \emptyset; \quad \text{No Global} \not\Rightarrow \mathcal{M} \neq \emptyset.$$

- This justifies the assumption $\mathcal{P}_0 \neq \varnothing$ (or equivalently $\mathcal{M} \neq \varnothing$).
- Next, we show that:

$$\text{No } \widetilde{\mathcal{H}} - \text{Robust Arbitrage (w.r.to } \sigma(\mathcal{P}, C_b)) \Longleftrightarrow \mathcal{M}_+ \neq \emptyset$$

No Arbitrage and M-polar sets

Let

$$\mathcal{N} := \{B \in \mathcal{F} \mid Q(B) = 0 \; \forall \; Q \in \mathcal{M}\}$$

be the family of polar sets of the class $\mathcal{M}$ of martingale measures.

Theorem

Let $(\Omega, \widetilde{\mathcal{F}}_T, (\widetilde{\mathcal{F}}_t)_{t \in I})$ be an opportune enlarged filtered space and Ω Polish. If $\mathcal{P}_0 \neq \varnothing$ then

$$\textit{No Open Arbitrage in } \widetilde{\mathcal{H}} \iff \mathcal{N} \textit{ does not contain open sets}$$

Essentially, under additional assumptions concerning the setting:

$$\text{No } \widetilde{\mathcal{H}} \text{ Arbitrage de la Classe } \mathcal{C} \iff \mathcal{N} \text{ does not contain sets of } \mathcal{C}$$

Summarizing

Theorem

Let $(\Omega, \widetilde{\mathcal{F}}_T, (\widetilde{\mathcal{F}}_t)_{t \in I})$ be an opportune enlarged filtered space and Ω Polish. If $\mathcal{P}_0 \neq \varnothing$ then TFAE

1. $NA(\mathcal{P}_+)$;
2. $\mathcal{M} \cap \mathcal{P}_+ \neq \varnothing$;
3. *No Open Arbitrage holds w.r.to the strategies in* $\widetilde{\mathcal{H}}$.
4. *No Robust Arbitrage holds w.r.to the strategies in* $\widetilde{\mathcal{H}}$.
5. *The market is feasible:* $\overline{\mathcal{P}_0}^{\sigma(\mathcal{P}, C_b)} = \mathcal{P}$

Arbitrage de la Classe C

Essentially, under appropriate assumptions, similar results can be obtained for the Arbitrage de la Classe $\mathcal{C}$, by replacing $\mathcal{P}_+$ with the probability measures in the class:

$$\mathcal{Q}(\mathcal{C}) := \{Q \in \mathcal{P}(\mathcal{C}) \mid Q(C) > 0 \ \forall C \in \mathcal{C}\}$$

Then the equivalent conditions are:

- $\mathcal{M} \cap \mathcal{Q}(\mathcal{C}) \neq \varnothing$;
- No Arbitrage de la Classe $\mathcal{C}$

When S is continuous with respect to the state variable

Corollary

Assume that $S_t : (\Omega, d) \to \mathbb{R}^d$ *is* **continuous** *for every* $t = 1, \ldots, T$. *Then*

$$\begin{aligned} \mathcal{M} \cap \mathcal{P}_+ \neq \varnothing \quad &\Leftrightarrow \quad \textit{No Robust Arbitrage holds w.r.to } \mathcal{H} \\ &\Leftrightarrow \quad \textit{No Pointwise Arbitrage holds w.r.to } \mathcal{H}. \end{aligned}$$

- This is the general version (multi-periods) of Riedel's result.
- No need to enlarge the filtration.

Technical Filtration

We build up a filtration enlargement $(\widetilde{\mathcal{F}}_t)_{t \in I}$ which follows directly from the market structure and

- It preserves the sets of martingale measures $\mathcal{M}(\widetilde{\mathbb{F}}) \rightleftarrows \mathcal{M}(\mathbb{F})$
  - The restriction of any $\widetilde{Q} \in \mathcal{M}(\widetilde{\mathbb{F}})$ to $\mathcal{F}_T$ belongs to $\mathcal{M}(\mathbb{F})$.
  - Any $Q \in \mathcal{M}(\mathbb{F})$ can be uniquely extended to an element of $\mathcal{M}(\widetilde{\mathbb{F}})$.
- There exists an aggregator $\widetilde{H} \in \widetilde{\mathcal{H}}$ for all P-a.s. arbitrage:
  - If $\widetilde{H} \in \widetilde{\mathcal{H}}$ with $\mathcal{V}^+_{\widetilde{H}} \in \mathcal{F}$ and there exists a probability $\widetilde{P} : \widetilde{\mathcal{F}}_T \to [0, 1]$ such that $\widetilde{P}(\mathcal{V}^+_{\widetilde{H}}) > 0$ then there exists a P–a.s. Arbitrage $H^P \in \mathcal{H}$, for P being the restriction of $\widetilde{P}$ on $\mathcal{F}$.

Simple example to explain the filtration enlargement

Let $(\Omega, \mathcal{F}) = (\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+))$. Consider a one period market with three assets: $S = [S^0, S^1, S^2]$, $\mathcal{F} = \mathcal{F}^S$, with $S_0 = [S_0^0, S_0^1, S_0^2] = [1, 1, 1]$, $S_1^0 = 1$ and

$$S_1^1 = \begin{cases} 2 & \omega \in \mathbb{R}^+ \setminus \mathbb{Q} \\ 1 & \omega \in \mathbb{Q}^+ \end{cases} \quad ; \quad S_1^2 = \begin{cases} \exp(\omega) & \omega \in \mathbb{R}^+ \setminus \mathbb{Q} \\ 0 & \omega = 0 \\ \exp(-\omega) & \omega \in \mathbb{Q}^+ \setminus \{0\} \end{cases}$$

1. There are no martingale measures for the process $S = [S^0, S^1, S^2]$:

$$\mathcal{M} = \varnothing$$

Indeed, if we denote by $\mathcal{M}_i$ the set of martingale measures for the i^{th} asset we have

$$\begin{aligned} \mathcal{M}_1 &= \{Q \in \mathcal{P} \mid Q(\mathbb{R}^+ \setminus \mathbb{Q}) = 0\} \\ \forall Q &\in \mathcal{M}_2,\ Q(\mathbb{R}^+ \setminus \mathbb{Q}) > 0 \end{aligned}$$

Example: continues

$$S_1^1 = \begin{cases} 2 & \omega \in \mathbb{R}^+ \setminus \mathbb{Q} \\ 1 & \omega \in \mathbb{Q}^+ \end{cases} \quad ; \quad S_1^2 = \begin{cases} \exp(\omega) & \omega \in \mathbb{R}^+ \setminus \mathbb{Q} \\ 0 & \omega = 0 \\ \exp(-\omega) & \omega \in \mathbb{Q}^+ \setminus \{0\} \end{cases}$$

2. If $V_0(H) = 0$ then the final value of the strategy $H = (-\alpha - \beta, \alpha, \beta) \in \mathbb{R}^3$ is

$$V_T(H) = \begin{cases} \alpha + \beta(\exp(\omega) - 1) & \omega \in \mathbb{R}^+ \setminus \mathbb{Q} \\ -\beta & \omega = 0 \\ \beta(\exp(-\omega) - 1) & \omega \in \mathbb{Q}^+ \setminus \{0\} \end{cases}.$$

3. Only the strategies $H \in \mathbb{R}^3$ having $\beta = 0$ and $\alpha \geq 0$ satisfy $V_T(H)(\omega) \geq 0$ for all $\omega \in \Omega$. For $\beta = 0$ and $\alpha > 0$, $v_H^+ = \mathbb{R}^+ \setminus \mathbb{Q}$ and therefore there are **No open Arbitrage** and **No global Arbitrage** (but $\mathcal{M} = \varnothing$).
4. This also justifies the assumption: $\mathcal{M} \neq \varnothing$.

$$S_1^1 = \begin{cases} 2 & \omega \in \mathbb{R}^+ \setminus \mathbb{Q} \\ 1 & \omega \in \mathbb{Q}^+ \end{cases} \quad ; \quad S_1^2 = \begin{cases} \exp(\omega) & \omega \in \mathbb{R}^+ \setminus \mathbb{Q} \\ 0 & \omega = 0 \\ \exp(-\omega) & \omega \in \mathbb{Q}^+ \setminus \{0\} \end{cases}$$

5. However, by fixing **any** probability P there **exists a P Arbitrage in the classical sense**, since the FTAP holds and $\mathcal{M} = \varnothing$. Indeed:
   1. If $P(\mathbb{R}^+ \setminus \mathbb{Q}) = 0$, then $\beta = -1$ ($\alpha = 0$) yield a P classical arbitrage, since $v_H^+ = \mathbb{Q}^+$ and $P(v_H^+) = 1$
   2. If $P(\mathbb{R}^+ \setminus \mathbb{Q}) > 0$ then $\beta = 0$ and $\alpha = 1$ yield a P classical arbitrage, since $v_H^+ = \mathbb{R}^+ \setminus \mathbb{Q}$ and $P(v_H^+) > 0$.

$$S_1^1 = \begin{cases} 2 & \omega \in \mathbb{R}^+ \setminus \mathbb{Q} \\ 1 & \omega \in \mathbb{Q}^+ \end{cases} \quad ; \quad S_1^2 = \begin{cases} \exp(\omega) & \omega \in \mathbb{R}^+ \setminus \mathbb{Q} \\ 0 & \omega = 0 \\ \exp(-\omega) & \omega \in \mathbb{Q}^+ \setminus \{0\} \end{cases}$$

5. However, by fixing **any** probability P there **exists a** P **Arbitrage in the classical sense**, since the FTAP holds and $\mathcal{M} = \varnothing$. Indeed:
    1. If $P(\mathbb{R}^+ \setminus \mathbb{Q}) = 0$, then $\beta = -1$ ($\alpha = 0$) yield a P classical arbitrage, since $v_H^+ = \mathbb{Q}^+$ and $P(v_H^+) = 1$
    2. If $P(\mathbb{R}^+ \setminus \mathbb{Q}) > 0$ then $\beta = 0$ and $\alpha = 1$ yield a P classical arbitrage, since $v_H^+ = \mathbb{R}^+ \setminus \mathbb{Q}$ and $P(v_H^+) > 0$.
6. Instead, by adopting the definition of a P-a.s. Arbitrage ($V_T(H)(\omega) \geq 0$ for all $\omega \in \Omega$ and $P(v_H^+) > 0$), there are two possibilities:
    1. If $P(\mathbb{R}^+ \setminus \mathbb{Q}) = 0$, **No** P**-a.s. Arbitrage** holds, since only the strategies $H \in \mathbb{R}^3$ having $\beta = 0$ and $\alpha \geq 0$ satisfies $V_T(H)(\omega) \geq 0$ for all $\omega \in \Omega$ and $v_H^+ = \mathbb{R}^+ \setminus \mathbb{Q}$.
    2. If $P(\mathbb{R}^+ \setminus \mathbb{Q}) > 0$, then $\beta = 0$ and $\alpha = 1$ **yield a** P**-a.s. arbitrage**, since $v_H^+ = \mathbb{R}^+ \setminus \mathbb{Q}$ and $P(v_H^+) > 0$.

Example: continues

$$S_1^1 = \begin{cases} 2 & \omega \in \mathbb{R}^+ \setminus \mathbb{Q} \\ 1 & \omega \in \mathbb{Q}^+ \end{cases} \quad ; \quad S_1^2 = \begin{cases} \exp(\omega) & \omega \in \mathbb{R}^+ \setminus \mathbb{Q} \\ 0 & \omega = 0 \\ \exp(-\omega) & \omega \in \mathbb{Q}^+ \setminus \{0\} \end{cases}$$

7 One way to remedy to 6.1) is to enlarge the filtration $\mathbb{F} \twoheadrightarrow \widetilde{\mathbb{F}}$ so that there will be an P-**a.s. Arbitrage:** We anticipate the $\mathcal{F}_1^S$ measurable P-**null set** $(\mathbb{R}^+ \setminus \mathbb{Q})$ to the initial sigma algebra $\mathcal{F}_0^S$. In this way the strategy $H = (1, 0, -1)1_{(\mathbb{R}^+ \setminus \mathbb{Q})^C}$ is predictable and satisfies $V_T(H) \geq 0$ for all $\omega \in \Omega$ (and $P(v_H^+) > 0$).

8 In general, the filtration enlargement $\mathbb{F} \twoheadrightarrow \widetilde{\mathbb{F}}$ that we adopt will only **anticipate $\mathcal{M}$-polar sets of one period.** It is intrinsic to the market and satisfies the above mentioned properties.

Alternative Definition

One other possibility to remedy to 6.1 would be to weaken the condition:

$$V_T(H)(\omega) \geq 0 \text{ for all } \omega \in \Omega.$$

as in the No classical Arbitrage:

$$P(V_T(H) < 0) = 0 \quad \Rightarrow \quad P(V_T(H) > 0) = 0$$

$$\mathcal{V}_H^- \text{ does contain } P \text{ positive sets} \quad \Rightarrow \quad \mathcal{V}_H^+ \text{ does not contain } P \text{ positive sets}$$

Definition (Modified No Arbitrage de la classe $\mathcal{C}$)

$$\mathcal{V}_H^- \text{ does not contain sets of } \mathcal{C} \Rightarrow \mathcal{V}_H^+ \text{ does not contain sets of } \mathcal{C}.$$

- However, the next example will show that this definition does not work.

$\mathcal{V}_H^+ := \{\omega \in \Omega \mid V_T(H)(\omega) > 0\}$ and $\mathcal{V}_H^- := \{\omega \in \Omega \mid V_T(H)(\omega) < 0\}$.

Example

No Open Arbitrage (in the modified definition):

$$\mathcal{V}_H^- \text{ does not contain an open set} \Rightarrow \mathcal{V}_H^+ \text{ does not contain an open set.}$$

Let $(\Omega, \mathcal{F}) = (\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+))$; one period market, one risky asset: $S_0 = 1$,

$$S_1 = \begin{cases} 2 & \omega \in [1, \infty) \\ 1 & \omega = [0, 1) \setminus \mathbb{Q} \\ 0 & \omega \in [0, 1) \cap \mathbb{Q} \end{cases}$$

Strategy: buy S :

- $\mathcal{V}_H^- = [0, 1) \cap \mathbb{Q}$ does not contain an open set,
- $\mathcal{V}_H^+ = [1, \infty)$ contains open sets
- Therefore there is an Open Arbitrage (in the modified definition).
- There are full support martingale measures, for example $Q([0, 1) \cap \mathbb{Q}) = Q([1, \infty)) = \frac{1}{2}$.

Heuristic Geometric approach

For $z \in \mathbb{R}^d$ consider the level sets:

$$\Sigma_{t-1}^z := \{\omega \in \Omega \mid S_{t-1}(\omega) = z\}$$

and the convex cone

$$K_t^z = co\left(conv\left(S_t(\omega) - z \mid \omega \in \Sigma_{t-1}^z\right)\right) \cup \{0\} \subseteq \mathbb{R}^d$$

- If $0 \in ri\left(K_t^z\right)$ we cannot apply hyperplane separating theorem to the convex sets $\{0\}$ and $ri\left(K_t^z\right)$, i.e. there is no $H \in \mathbb{R}^d$ s.t. $H(S_t(\omega) - z) \geq 0$ for all $\omega \in \Sigma_{t-1}^z$ with strict inequality for some ω.
- $0 \in ri\left(K_t^z\right)$ if and only if No Pointwise Arbitrage are possible on the set Σ_{t-1}^z, since a trading strategy on Σ_{t-1}^z with a non-zero payoff always yields both positive and negative outcomes.
- In this case, the level set is not suitable for the construction of a pointwise arbitrage opportunity and sets with this property are naturally important for the construction of a martingale measure.
- If $0 \notin ri\left(K_t^z\right)$, we wish to identify those subset of Σ_{t-1}^z that retain this property.

Heuristic Geometric Approach

$$\Sigma_{t-1}^{z} = \{\omega \in \Omega \mid S_{t-1} = z\} \in \mathcal{F}_t \quad \text{for } z \in \mathbb{R}^d.$$
$$K_t^z = co\left(conv\left(S_t(\omega) - z \mid \omega \in \Sigma_{t-1}^z\right)\right) \cup \{0\} \subseteq \mathbb{R}^d$$

Lemma

If $0 \notin ri\left(K_t^z\right)$ then there exist $\beta \in \{1, \ldots, d\}$, $B_{t,z}^1, \ldots, B_{t,z}^{\beta}$ and $H_{t,z}^1, \ldots, H_{t,z}^{\beta}$ with $B_{t,z}^i \in \mathcal{F}_t$ non empty and $H_{t,z}^i \in \mathbb{R}^d$ such that:

- $B_{t,z}^{\star} = \Sigma_{t-1}^{z} \setminus \left(\cup_{j=1}^{\beta} B_{t,z}^{j}\right)$
- $B_{t,z}^i \cap B_{t,z}^j = \varnothing$ *if* $i \neq j$;
- $\forall i \leq \beta,\ H_{t,z}^i(S_t(\omega) - z) > 0$ *for all* $\omega \in B_{t,z}^i$ *and* $H_{t,z}^i(S_t(\omega) - z) \geq 0$ *for all* $\omega \in \cup_{j=i}^{\beta} B_{t,z}^j \cup B_{t,z}^{\star}$.
- $\forall H \in \mathbb{R}^d$ *s.t.* $H(S_t(\omega) - z) \geq 0$ *on* $B_{t,z}^{\star}$ *we have* $H(S_t(\omega) - z) = 0$ *on* $B_{t,z}^{\star}$.

$\beta = \beta_{t,z}$ will depend on t and z.

Interpretation

The difference between the sets $B^i_{t,z}$ and $B^*_{t,z}$:

- Restricted to the time interval $[t-1, t]$, a probability measure whose mass is concentrated on $B^*_{t,z}$ admits an equivalent martingale measure
- For those probabilities that assign positive mass to at least one $B^i_{t,z}$ an arbitrage opportunity can be constructed.

Example of separation of atoms

Let $(\Omega, \mathcal{F}) = (\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+))$. Consider the market $S_0 = [1, 1, 1]$ and $S_t = [S_t^1, S_t^2, S_t^3]$ with

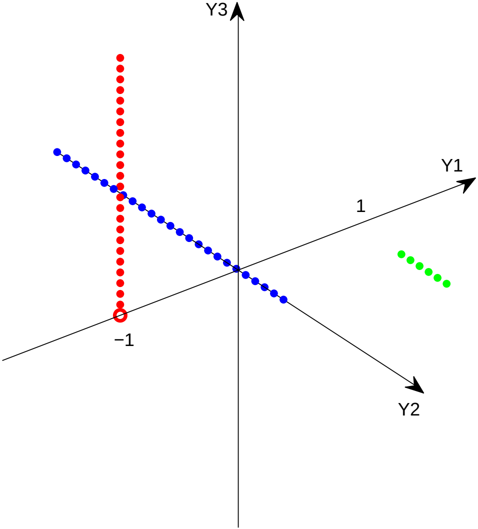
$$S_1^1(\omega) = \begin{cases} 0 & \omega \in \mathbb{R}^+ \setminus \mathbb{Q} \\ 1 & \omega \in \mathbb{Q} \cap (1/2, +\infty) \\ 2 & \omega \in \mathbb{Q} \cap (0, 1/2) \end{cases} \quad S_1^2(\omega) = \begin{cases} 1 & \omega \in \mathbb{R}^+ \setminus \mathbb{Q} \\ \omega^2 & \omega \in \mathbb{Q} \cap (1/2, +\infty) \\ \omega^2 & \omega \in \mathbb{Q} \cap (0, 1/2) \end{cases}$$

$$S_1^3(\omega) = \begin{cases} 1 + \omega^2 & \omega \in \mathbb{R}^+ \setminus \mathbb{Q} \\ 1 & \omega \in \mathbb{Q} \cap (1/2, +\infty) \\ 1 & \omega \in \mathbb{Q} \cap (0, 1/2) \end{cases}$$

Example of separation of atoms

Recall $Y_i := S_1^i - S_0^i$

In this example:
$B^1 = \mathbb{R}^+ \setminus \mathbb{Q}$
$B^2 = \mathbb{Q} \cap (0, 1/2)$
$B^* = \mathbb{Q} \cap (1/2, +\infty)$



The sets Bi are M-polar sets

Lemma

Let $Q \in \mathcal{M}$ and $\mathcal{F}_t^Q$ be the completion of $\mathcal{F}_t$ and $\overline{Q} : \mathcal{F}_t^Q \to [0,1]$ be the unique extension on $\mathcal{F}_t^Q$ of Q. If

$$\mathfrak{B}_t := \bigcup_{z \in \mathbb{R}^d} \left\{ \cup_{i=1}^{\beta} B_{t,z}^i \right\}$$

for $B_{t,z}^i$ given in the previous Lemma then $\mathfrak{B}_t \in \mathcal{F}_t^Q$ and

$$\overline{Q}(\mathfrak{B}_t) = 0.$$

$$K_t^z := co\left(conv\left(S_t(\omega) - z \mid \omega \in \Sigma_{t-1}^z\right)\right) \cup \{0\}.$$

For any Σ_{t-1}^z such that in $0 \notin ri(K_t^z)$ consider the sets $B_{t,z}^i$ and:

$$\mathcal{N}_t := \left\{ E \in 2^{\Omega} \mid E = \bigcup \{B_{t,z}^i\} \right\}$$

Filtration

We consider the enlarged filtration $\{\widetilde{\mathcal{F}}_t\}_{t\in I}$ given by

$$\widetilde{\mathcal{F}}_t = \mathcal{F}_t \vee \bigcup_{s=1}^{t+1} \mathcal{N}_s \quad \text{for } t < T \text{ and } \widetilde{\mathcal{F}}_T = \mathcal{F}_t \vee \bigcup_{s=1}^{T} \mathcal{N}_s.$$

Then the universal arbitrage (the universal aggregator) is given by:

$$H_t(\omega) = \sum_{z\in\mathbb{R}^d} \sum_{i=1}^{\beta_{t,z}} H^i_{t,z} \mathbf{1}_{B^i_{t,z}}(\omega)$$

where $H^i_{t,z}$ are constructed in the previous procedure.

Thank you for the attention