Arbitrage of the first kind and filtration enlargements in semimartingale financial models

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(based on a joint work with C. Fontana and C. Kardaras)
The problem:

- Consider a market without arbitrage profits.
- Suppose some agents have additional information.
- Can they use this information to realize arbitrage profits?
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Mathematically:

The market: \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, S)\), with \(\mathbb{F}\) satisfying the usual conditions, 
\[ S = (S^i)_{i=1,...,d} \text{ non-negative semimartingale}, \quad S^0 \equiv 1. \]

Additional information:
- progressive enlargement of filtration (with any random time)
- initial enlargement of filtration

Arbitrage profits: ...(some motivation first)
The basic example

- Let $W$ be a standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{F}^W, \mathbb{P})$.
- Let $S$ represent the discounted price of an asset and be given by

  \[ S_t = \exp \left( \sigma W_t - \frac{1}{2} \sigma^2 t \right), \quad \sigma > 0 \text{ given.} \]

- Let $S^*_t := \sup\{S_u, \ u \leq t\}$ and define the random time

  \[ \tau := \sup\{t : S_t = S^*_\infty\} = \sup\{t : S_t = S^*_t\} \]

- An agent with information $\tau$ can follow the arbitrage strategy

  "buy at $t = 0$ and sell at $t = \tau"
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- An agent with information $\tau$ can follow the arbitrage strategy
  
  "buy at $t = 0$ and sell at $t = \tau"$

Remark. Here $\tau$ is an honest time: $\forall \ t \geq 0 \ \exists \ \xi_t \ \mathcal{F}^W_t$-measurable s.t. $\tau = \xi_t$ on $\{ \tau \leq t \}$ (e.g., $\xi_t := \sup \{ u \leq t : S_u = \sup_{r \leq t} S_r \}$).

Different notions of arbitrage

Admissible wealth processes $\mathcal{X}(\mathbb{F}, S) :$ class of all *non-negative* processes of the type $X^{x, H} := x + \int_0^t H_t dS_t.$

We recall the notions of:

- **arbitrage:** $\exists X^{1, H} \in \mathcal{X}(\mathbb{F}, S)$ s.t. $\mathbb{P}[X^{1, H}_\infty \geq 1] = 1,$ $\mathbb{P}[X^{1, H}_\infty > 1] > 0.$
  If such strategies do not exist we say that NA($\mathbb{F}, S$) holds.

- **free lunch with vanishing risk:** $\exists \epsilon > 0,$ $0 \leq \delta_n \uparrow 1,$ $X^{1, H^n} \in \mathcal{X}(\mathbb{F}, S)$ s.t. $\mathbb{P}[X^{1, H^n}_\infty > \delta_n] = 1,$ $\mathbb{P}[X^{1, H^n}_\infty > 1 + \epsilon] \geq \epsilon.$
  If such strategies do not exist we say that NFLVR($\mathbb{F}, S$) holds.

- **arbitrage of the first kind:** $\exists \xi \geq 0$ with $\mathbb{P}[\xi > 0] > 0$ s.t. for all $x > 0,$ $\exists X \in \mathcal{X}(\mathbb{F}, S)$ with $X_0 = x$ s.t. $\mathbb{P}[X_\infty \geq \xi] = 1.$
  If such strategies do not exist we say that NA1($\mathbb{F}, S$) holds.

**Remark.** NA1 (Kardaras, 2010) $\iff$ BK (Kabanov, 1997) $\iff$ NUPBR (Karatzas, Kardaras 2007)
NFLVR $\iff$ NA + NA1
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NFLVR $\iff$ $\exists$ equivalent local martingale measure for $S$
Martingale measures and deflators

- NFLVR $\iff$ NA $+$ NA1

- NFLVR $\iff$ $\exists$ equivalent local martingale measure for $S$

- NA1 $\iff$ $\exists$ supermartingale deflator (Karatzas, Kardaras’07):
  $Y > 0$, $Y_0 = 1$ s.t. $YX$ is a supermartingale $\forall X \in \mathcal{X}$
  $\iff$ $\exists$ local martingale deflator (Takaoka, Schweizer’13, Song’13):
  $Y > 0$, $Y_0 = 1$ s.t. $YX$ is a local martingale $\forall X \in \mathcal{X}$

- NA1 $\iff$ $\exists$ readable local martingale deflator (A.F.K.’14):
  $Y$ local martingale deflator s.t. $1/Y \in \mathcal{X}$ (up to $Q \sim P$)
Why NA1? - Let me try to convince you

- As seen in the basic example, NA and NFLVR easily fail under additional information.

- NA1 is the minimal condition in order to proceed with utility maximization.

- NA1 is stable under change of numéraire.

- NA1 is equivalent to the existence of a numéraire portfolio $X^\ast$ (= growth optimal portfolio = log optimal portfolio), in which $1/X^\ast$ is a supermartingale deflator.
Why NA1? - Let me try to convince you

- As seen in the basic example, **NA and NFLVR easily fail under additional information.**
- Whereas when an **arbitrage** exists we are in general not able to spot it, when an **arbitrage of the first kind** exists we are able to construct (and hence exploit) it (NA1 is completely characterized in terms of the characteristic triplet of $S$).
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- As seen in the basic example, **NA and NFLVR easily fail under additional information**.
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- NA1 is the minimal condition in order to proceed with utility maximization.
- NA1 is stable under change of numéraire.
- NA1 is equivalent to the existence of a numéraire portfolio $X^*$ ($= \text{growth optimal portfolio} = \text{log optimal portfolio}$), in which case $1/X^*$ is a supermartingale deflator.
Some related work (NA1 preservation)

- Fontana, Jeanblanc, Song 2013: S continuous, PRP, \( \tau \) honest and avoids all \( \mathbb{F} \)-stopping times, NFLVR in the original market. Then in the enlarged market:
  - on \([0, \infty)\): NA1, NA and NFLVR all fail;
  - on \([0, \tau]\): NA and NFLVR fail, but NA1 holds.

- Kreher 2014:
  all \( \mathbb{F} \)-martingales are continuous, \( \tau \) avoids all \( \mathbb{F} \)-stopping times, NFLVR in the original market.

- Aksamit, Choulli, Deng, Jeanblanc 2013:
  using optional stochastic integral, \((S \text{ quasi-left-continuous})\).
Let $\tau$ be a random time (= positive, finite, $\mathcal{F}$-measurable r.v.).

Consider the progressively enlarged filtration $\mathcal{G} = (\mathcal{G}_t)_{t \in \mathbb{R}_+}$, where

$$\mathcal{G}_t := \{ B \in \mathcal{F} \mid B \cap \{\tau > t\} = B_t \cap \{\tau > t\} \text{ for some } B_t \in \mathcal{F}_t \}.$$
Let $\tau$ be a random time (= positive, finite, $\mathcal{F}$-measurable r.v.).

Consider the progressively enlarged filtration $\mathcal{G} = (\mathcal{G}_t)_{t \in \mathbb{R}_+}$,

$$\mathcal{G}_t := \{ B \in \mathcal{F} \mid B \cap \{ \tau > t \} = B_t \cap \{ \tau > t \} \text{ for some } B_t \in \mathcal{F}_t \}.$$

Jeulin-Yor theorem ensures that $\mathcal{H}'$-hypothesis holds up to $\tau$: every $\mathcal{F}$-semimartingale remains a $\mathcal{G}$-semimartingale up to time $\tau$ (in particular $S^\tau$ is a $\mathcal{G}$-semimartingale).
Consider the two $\mathbb{F}$-supermartingales associated to $\tau$:

\[
Z_t := \mathbb{P}[\tau > t \mid \mathcal{F}_t], \quad \tilde{Z}_t := \mathbb{P}[\tau \geq t \mid \mathcal{F}_t]
\]

($Z =$ Azéma supermartingale associated to $\tau$)
Our main tools

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($Z = \text{Azéma supermartingale}$ associated to $\tau$)

Let $A$ be the $\mathbb{F}$-dual optional projection of $\mathbb{1}_{[\tau, \infty[}$, then

$$\Delta A_\sigma = \tilde{Z}_\sigma - Z_\sigma = \mathbb{P}[\tau = \sigma \mid \mathcal{F}_\sigma] \quad \text{for all } \mathbb{F}\text{-stopping times } \sigma.$$
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Define the stopping time \( \zeta := \inf \{ t \in \mathbb{R}_+ \mid Z_t = 0 \} \geq \tau. \)
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  \[ Z_t := \mathbb{P}[\tau > t \mid \mathcal{F}_t], \quad \tilde{Z}_t := \mathbb{P}[\tau \geq t \mid \mathcal{F}_t] \]
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- Define the stopping time $\zeta := \inf \{ t \in \mathbb{R}_+ \mid Z_t = 0 \} \geq \tau$.

- Define $\Lambda := \{ \zeta < \infty, Z_{\zeta-} > 0, \Delta A_\zeta = 0 \} \in \mathcal{F}_\zeta$
  \[ = \text{set where } Z \text{ jumps to zero after } \tau \]}
Our main tools

- Consider the two $\mathbb{F}$-supermartingales associated to $\tau$: 
  \[
  Z_t := \mathbb{P}[\tau > t \mid \mathcal{F}_t], \quad \tilde{Z}_t := \mathbb{P}[\tau \geq t \mid \mathcal{F}_t]
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- Let $A$ be the $\mathbb{F}$-dual optional projection of $\mathbb{I}_{[\tau, \infty]}$, then 
  \[
  \Delta A_\sigma = \tilde{Z}_\sigma - Z_\sigma = \mathbb{P}[\tau = \sigma \mid \mathcal{F}_\sigma] \quad \text{for all $\mathbb{F}$-stopping times $\sigma$}.
  \]

- Define the stopping time $\zeta := \inf \{ t \in \mathbb{R}_+ \mid Z_t = 0 \} \geq \tau$.

- Define $\Lambda := \{ \zeta < \infty, \ Z_\zeta^- > 0, \ \Delta A_\zeta = 0 \} \in \mathcal{F}_\zeta$

  = set where $Z$ jumps to zero after $\tau$

- and define

  \[
  \eta := \zeta \mathbb{I}_\Lambda + \infty \mathbb{I}_{\Omega \setminus \Lambda}
  \]

Note that $\tau < \eta$; $\eta = \text{time when $Z$ jumps to zero after $\tau$}$. 
Representation pair associated with $\tau$

**Theorem** (Itô, Watanabe 1965, Kardaras 2014).
The Azéma supermartingale $Z$ admits the following multiplicative decomposition:

$$Z = L(1 - K),$$

where:

- $L$ is a nonnegative $\mathbb{F}$-local martingale with $L_0 = 1$,
- $K$ is a nondecreasing $\mathbb{F}$-adapted process with $0 \leq K \leq 1$,
- for any nonnegative optional processes $V$ on $(\Omega, \mathbb{F})$,

$$\mathbb{E}[V_\tau] = \mathbb{E} \left[ \int_{\mathbb{R}^+} V_t \, L_t \, dK_t \right].$$

Together with the stopping time $\eta$, the local martingale $L$ will play a main role in our results.
Back to the basic example

Asset price process: \( S_t = \exp \left( \sigma W_t - \frac{1}{2} \sigma^2 t \right) \)

Random time: \( \tau := \sup\{ t : S_t = S^*_\infty \} \)

In this case
\[
Z_t = \mathbb{P}[\tau > t \mid \mathcal{F}_t] = \frac{S_t}{S^*_t}
\]

Therefore:

\( \eta = \infty \)

\( L = S \)

\( Y := 1/L^\tau = 1/S^\tau \) is a local martingale deflator for \( S^\tau \) in \( \mathbb{G} \).

\( \Rightarrow \) NA1 holds while NA and NFLVR fail.
Remember: \( \eta \) is the time when \( Z \) jumps to zero after \( \tau \).

**Proposition.** Let \( X \) be a nonnegative \( \mathbb{F} \)-local martingale such that \( X = 0 \) on \([\eta, \infty[\). Then \( X^\tau / L^\tau \) is a \( \mathbb{G} \)-local martingale.

\( \triangleright \) The main tool in the proof of the proposition is the multiplicative decomposition of \( Z \).
Remember: $\eta$ is the time when $Z$ jumps to zero after $\tau$.

**Proposition.** Let $X$ be a nonnegative $\mathbb{F}$-local martingale such that $X = 0$ on $[\eta, \infty]$. Then $X^\tau/L^\tau$ is a $\mathbb{G}$-local martingale.

The main tool in the proof of the proposition is the multiplicative decomposition of $Z$.

As an immediate consequence we have the following

**Key-Proposition.** Suppose there exists a local martingale deflator $M$ for $S$ in $\mathbb{F}$ such that $M = 0$ on $[\eta, \infty]$. Then $M^\tau/L^\tau$ is a local martingale deflator for $S^\tau$ in $\mathbb{G}$. 
To have preservation of the NA1 property, given a deflator for \( S \) in \( \mathbb{F} \), we want to “kill it” from \( \eta \) on.

We will do it with the help of the following lemma.

**Lemma.** Let \( D \) be the \( \mathbb{F} \)-predictable compensator of \( \mathbb{I}_{[\eta, \infty]} \). Then:
- \( \Delta D < 1 \) \( \mathbb{P} \)-a.s. (\( \Rightarrow \mathcal{E}(-D) > 0 \) and nonincreasing);
- \( \mathcal{E}(-D)^{-1}\mathbb{I}_{[0, \eta]} \) is a local martingale on \((\Omega, \mathbb{F}, \mathbb{P})\).

Main idea: for any predictable time \( \sigma \) on \((\Omega, \mathbb{F})\),
\[
\Delta D_{\sigma} = \mathbb{P} [\eta = \sigma \mid \mathcal{F}_{\sigma-}] < 1 \quad \text{on} \quad \{\sigma < \infty\}.
\]
Theorem (one fixed $S$). Suppose that $\mathbb{P}[\eta < \infty, \Delta S_\eta \neq 0] = 0$. If $\text{NA1}(\mathbb{F}, S)$ holds, then $\text{NA1}(\mathbb{G}, S^\tau)$ holds.

That is: $S$ does not jump when $Z$ jumps to zero.
Theorem (one fixed $S$). Suppose that $\mathbb{P} [\eta < \infty, \Delta S_\eta \neq 0] = 0$. If NA1($\mathcal{F}, S$) holds, then NA1($\mathcal{G}, S^\tau$) holds.

That is: $S$ does not jump when $Z$ jumps to zero.

Remark. Condition $\mathbb{P} [\eta < \infty, \Delta S_\eta \neq 0] = 0$ is equivalent to evanescence of the set $\{Z_- > 0, \tilde{Z} = 0, \Delta S \neq 0\}$. (See Aksamit et al. (2013), where $S$ is a quasi-left-continuous local martingale.) → see Monique’s talk
Recall: $D$ is the $\mathbb{F}$-predictable compensator of $\mathbb{I}_{[\eta, \infty[}$. 

- $\text{NA1}(\mathbb{F}, S) \Rightarrow \exists \hat{X} \in \mathcal{X}(\mathbb{F}, S)$ s.t. $Y := (1/\hat{X})$ is a local martingale deflator for $S$ in $\mathbb{F}$ ($\Rightarrow \Delta Y = 0$ when $\Delta S = 0$).

- In order to apply the Key-Proposition, we need a deflator for $S$ in $\mathbb{F}$ that vanishes on the set $[\eta, \infty[$.
Proof of the theorem

Recall: $D$ is the $\mathbb{F}$-predictable compensator of $\mathbb{I}_{[\eta, \infty]}$.

- $\text{NA1}(\mathbb{F}, S) \Rightarrow \exists \hat{X} \in \mathcal{X}(\mathbb{F}, S) \text{ s.t. } Y := (1/\hat{X})$ is a local martingale deflator for $S$ in $\mathbb{F}$ ($\Rightarrow \Delta Y = 0$ when $\Delta S = 0$).

- In order to apply the Key-Proposition, we need a deflator for $S$ in $\mathbb{F}$ that vanishes on the set $[\eta, \infty]$.

- Let $M := Y \mathcal{E}(-D)^{-1} \mathbb{I}_{[0, \eta]}$ ($\Rightarrow \{M > 0\} = [0, \eta]$).

- By the Lemma, $MS - \left[ \mathcal{E}(-D)^{-1} \mathbb{I}_{[0, \eta]}, YS \right] \mathbb{F}$-local martingale.
Proof of the theorem

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- $\text{NA1}(\mathbb{F}, S) \Rightarrow \exists \hat{X} \in \mathcal{X}(\mathbb{F}, S)$ s.t. $Y := (1/\hat{X})$ is a local martingale deflator for $S$ in $\mathbb{F}$ ($\Rightarrow \Delta Y = 0$ when $\Delta S = 0$).

- In order to apply the Key-Proposition, we need a deflator for $S$ in $\mathbb{F}$ that vanishes on the set $[\eta, \infty]$.

- Let $M := Y \mathcal{E}(\mathbb{D})^{-1} \mathbb{I}_{[0, \eta]}$ ($\Rightarrow \{M > 0\} = [0, \eta]$).

- By the Lemma, $MS - [\mathcal{E}(\mathbb{D})^{-1} \mathbb{I}_{[0, \eta]}, YS]$ $\mathbb{F}$-local martingale.

- We want $M$ to be a deflator for $S$ in $\mathbb{F}$, so we need to show that the quadratic covariation part is an $\mathbb{F}$-local martingale.

- $\Delta S_\eta = 0 \Rightarrow \Delta (YS)_\eta = 0 \Rightarrow [.., ..] = [\mathcal{E}(\mathbb{D})^{-1}, YS]$, which is indeed an $\mathbb{F}$-local martingale.
Theorem (general stability). TFAE:

1) for any $S$ s.t. $\text{NA1}(\mathbb{F}, S)$ holds, $\text{NA1}(\mathbb{G}, S^\tau)$ holds;

2) $\eta = \infty$ $\mathbb{P}$-a.s.;

3) For every nonnegative local martingale $X$ on $(\Omega, \mathbb{F}, \mathbb{P})$, the process $X^\tau/L^\tau$ is a local martingale on $(\Omega, \mathbb{G}, \mathbb{P})$;

4) The process $1/L^\tau$ is a local martingale on $(\Omega, \mathbb{G}, \mathbb{P})$.

Remark. Condition 2) is equivalent to evanescence of the set $\{Z_- > 0, \tilde{Z} = 0\} = \{Z_- > 0, Z = 0, \Delta A = 0\}$. (See Aksamit et al. (2013).)
Proof of the theorem

2) $\Rightarrow$ 1): from previous Theorem.

1) $\Rightarrow$ 2): suppose $\mathbb{P} [\eta < \infty] > 0$. Define

$$S := \mathcal{E}(-D)^{-1}\mathbb{1}_{[0, \eta]}.$$

Then $S$ is a $\mathbb{F}$-local martingale, and $S^\tau$ is nondecreasing with $\mathbb{P} [S^\tau > 1] > 0$. Hence NA1($\mathbb{F}, S$) holds, but NA1($\mathbb{G}, S^\tau$) fails.

2) $\Rightarrow$ 3): from the Proposition.

3) $\Rightarrow$ 4): trivial.

4) $\Rightarrow$ 2): uses properties of the processes $L$ and $K$ appearing in the multiplicative decomposition of $Z$. 
On the $\mathcal{H}'$-hypothesis

**Proposition.** Let $X$ be a nonnegative $\mathbb{F}$-supermartingale. Then, the process $X^\tau/L^\tau$ is a $\mathbb{G}$-supermartingale.

**Remark.** This can be used to establish that for any semimartingale $X$ on $(\Omega, \mathbb{F}, \mathbb{P})$, the process $X^\tau$ is a semimartingale on $(\Omega, \mathbb{G}, \mathbb{P})$. Indeed:

- By the Proposition, $\forall X$ nonnegative bounded $\mathbb{F}$-local martingale $\Rightarrow X^\tau/L^\tau$ and $1/L^\tau$ are $\mathbb{G}$-semimartingales $\Rightarrow X^\tau$ is a $\mathbb{G}$-semimartingale.
- From the semimartingale decomposition + localisation, same result for any $\mathbb{F}$-semimartingale $X$. 
Example: failure of $\text{NA1}(\mathcal{G}, S^\tau)$

- Consider $\zeta : \Omega \mapsto \mathbb{R}_+$ such that $\mathbb{P}[\zeta > t] = \exp(-t), \forall \ t \in \mathbb{R}_+$.
- Let $\mathcal{F} = (\mathcal{F}_t)_{t \in \mathbb{R}_+}$ be the smallest filtration that satisfies the usual hypotheses and makes $\zeta$ a stopping time.
- Define $\tau := \zeta/2$. 
Example: failure of $\text{NA1}(\mathcal{G}, S^\tau)$

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- Define $\tau := \zeta/2$.

- Note that $Z_t := \mathbb{P}[\tau > t|\mathcal{F}_t] = \exp(-t)\mathbb{I}_{\{t < \zeta\}}$ for all $t \in \mathbb{R}_+$.
- Note that $\zeta = \inf\{t \geq 0 \mid Z_t = 0\} =: \eta < \infty \ \mathbb{P}\text{-a.s.}$
- The $\mathbb{F}$-pred. comp. of $\mathbb{I}_{[\eta, \infty]}$ is $D := \eta \wedge t)_{t \in \mathbb{R}_+}$. 

$\text{S} = \mathbb{E}(\frac{1}{D} - 1_{[0, \eta]}$,...
Example: failure of \( \text{NA1}(\mathcal{G}, S^\tau) \)

Consider \( \zeta : \Omega \mapsto \mathbb{R}_+ \) such that \( \mathbb{P}[\zeta > t] = \exp(-t), \forall \ t \in \mathbb{R}_+ \).

Let \( \mathcal{F} = (\mathcal{F}_t)_{t \in \mathbb{R}_+} \) be the smallest filtration that satisfies the usual hypotheses and makes \( \zeta \) a stopping time.

Define \( \tau := \zeta/2 \).

Note that \( Z_t := \mathbb{P}[\tau > t|\mathcal{F}_t] = \exp(-t)\mathbb{I}_{\{t < \zeta\}} \) for all \( t \in \mathbb{R}_+ \).

Note that \( \zeta = \inf \{ t \geq 0 \mid Z_t = 0 \} =: \eta < \infty \) \( \mathbb{P} \)-a.s.

The \( \mathcal{F} \)-pred. comp. of \( \mathbb{I}_{[\eta, \infty]} \) is \( D := (\eta \wedge t)_{t \in \mathbb{R}_+} \).

\( S := \mathcal{E}(-D)^{-1}\mathbb{I}_{[0, \eta]} = \exp(D)\mathbb{I}_{[0, \eta]} \), that is, \( S_t = \exp(t)\mathbb{I}_{\{t < \zeta\}} \).

\( S \) nonnegative \( \mathcal{F} \)-martingale \( \Rightarrow \) \( \text{NA1}(\mathcal{F}, S) \).

But \( S \) is strictly increasing up to \( \tau \) \( \Rightarrow \) \( \text{NA1}(\mathcal{G}, S^\tau) \) fails.
Conclusions

- We provide a simple and general condition for preservation of NA1 under filtration enlargement for any fixed semimartingale model.
- We obtain a characterization of NA1 stability under filtration enlargement in a robust context, that is, for all possible semimartingale models.
- We use easy techniques.
- We obtain parallel results under progressive and initial enlargements.
Conclusions

- We provide a simple and general condition for preservation of NA1 under filtration enlargement for any fixed semimartingale model.

- We obtain a characterization of NA1 stability under filtration enlargement in a robust context, that is, for all possible semimartingale models.

- We use easy techniques.

- We obtain parallel results under progressive and initial enlargements.

Thank you for your attention!
Let $\mathbf{J}$ be an $\mathcal{F}$-measurable random variable taking values in a Lusin space $(E, \mathcal{B}_E)$, where $\mathcal{B}_E$ denotes the Borel $\sigma$-field of $E$.

Let $\mathcal{G} = (\mathcal{G}_t)_{t \in \mathbb{R}^+}$ be the right-continuous augmentation of the filtration $\mathcal{G}^0 = (\mathcal{G}^0_t)_{t \in \mathbb{R}^+}$ defined by

$$\mathcal{G}^0_t := \mathcal{F}_t \vee \sigma(\mathbf{J}), \quad t \in \mathbb{R}^+.$$
Initial enlargement of filtrations

- Let $J$ be an $\mathcal{F}$-measurable random variable taking values in a Lusin space $(E, \mathcal{B}_E)$, where $\mathcal{B}_E$ denotes the Borel $\sigma$-field of $E$.

- Let $\mathcal{G} = (\mathcal{G}_t)_{t \in \mathbb{R}_+}$ be the right-continuous augmentation of the filtration $\mathcal{G}^0 = (\mathcal{G}_t^0)_{t \in \mathbb{R}_+}$ defined by

$$
\mathcal{G}_t^0 := \mathcal{F}_t \vee \sigma(J), \quad t \in \mathbb{R}_+.
$$

- Let $\gamma : \mathcal{B}_E \mapsto [0, 1]$ be the law of $J$ ($\gamma [B] = \mathbb{P} [J \in B], B \in \mathcal{B}_E$).

- For all $t \in \mathbb{R}_+$, let $\gamma_t : \Omega \times \mathcal{B}_E \mapsto [0, 1]$ be a regular version of the $\mathcal{F}_t$-conditional law of $J$. 

Jacod's hypothesis.

We assume $\gamma_t \ll \gamma_{\mathbb{P}}$-a.s., $t \in \mathbb{R}_+$.

This ensures the $H'$-hypothesis and that we can apply Stricker & Yor calculus with one parameter ($L_1(\Omega, \mathcal{F}, \mathbb{P})$ separable).
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$$\mathcal{G}_t^0 := \mathcal{F}_t \vee \sigma(J), \quad t \in \mathbb{R}^+.$$  

Let $\gamma : \mathcal{B}_E \mapsto [0, 1]$ be the law of $J$ ($\gamma [B] = \mathbb{P} [J \in B], B \in \mathcal{B}_E$).

For all $t \in \mathbb{R}^+$, let $\gamma_t : \Omega \times \mathcal{B}_E \mapsto [0, 1]$ be a regular version of the $\mathcal{F}_t$-conditional law of $J$.

**Jacod’s hypothesis.** We assume

$$\gamma_t \ll \gamma \quad \mathbb{P}\text{-a.s.,} \quad t \in \mathbb{R}^+.$$  

This ensures the $\mathcal{H}'$-hypothesis and that we can apply Stricker& Yor calculus with one parameter ($\mathbb{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ separable).
Our main tools

\(\mathcal{O}(\mathbb{F})\) (resp. \(\mathcal{P}(\mathbb{F})\)) is the \(\mathbb{F}\)-optional (resp. pred.) \(\sigma\)-field on \(\Omega \times \mathbb{R}_+\).

**Lemma.** There exists a \(\mathcal{B}_E \otimes \mathcal{O}(\mathbb{F})\)-measurable function 

\[
E \times \Omega \times \mathbb{R}_+ \ni (x, \omega, t) \mapsto p^x_t(\omega) \in [0, \infty),
\]

całdąg in \(t \in \mathbb{R}_+\) s.t.:

- \(\forall t \in \mathbb{R}_+, \gamma_t(dx) = p^x_t \gamma(dx)\) holds \(\mathbb{P}\)-a.s;
- \(\forall x \in E, p^x = (p^x_t)_{t \in \mathbb{R}_+}\) is a martingale on \((\Omega, \mathbb{F}, \mathbb{P})\).
Our main tools

\( \mathcal{O}(\mathbb{F}) \) (resp. \( \mathcal{P}(\mathbb{F}) \)) is the \( \mathbb{F} \)-optional (resp. pred.) \( \sigma \)-field on \( \Omega \times \mathbb{R}_+ \)

**Lemma.** There exists a \( \mathcal{B}_E \otimes \mathcal{O}(\mathbb{F}) \)-measurable function 
\( E \times \Omega \times \mathbb{R}_+ \ni (x, \omega, t) \mapsto p^x_t(\omega) \in [0, \infty) \), càdlàg in \( t \in \mathbb{R}_+ \) s.t.:

- \( \forall t \in \mathbb{R}_+ \), \( \gamma_t(dx) = p^x_t \gamma(dx) \) holds \( \mathbb{P} \)-a.s;
- \( \forall x \in E, \ p^x = (p^x_t)_{t \in \mathbb{R}_+} \) is a martingale on \( (\Omega, \mathcal{F}, \mathbb{P}) \).

▷ For every \( x \in E \) define

\[
\zeta^x := \inf \{ t \in \mathbb{R}_+ \mid p^x_t = 0 \}.
\]
Our main tools

\( \mathcal{O}(\mathbb{F}) \) (resp. \( \mathcal{P}(\mathbb{F}) \)) is the \( \mathbb{F} \)-optional (resp. pred.) \( \sigma \)-field on \( \Omega \times \mathbb{R}_+ \).

**Lemma.** There exists a \( \mathcal{B}_E \otimes \mathcal{O}(\mathbb{F}) \)-measurable function \( E \times \Omega \times \mathbb{R}_+ \ni (x, \omega, t) \mapsto p_t^x(\omega) \in [0, \infty) \), càdlàg in \( t \in \mathbb{R}_+ \) s.t.:
- \( \forall t \in \mathbb{R}_+ \), \( \gamma_t(dx) = p_t^x(\gamma(dx)) \) holds \( \mathbb{P} \)-a.s;
- \( \forall x \in E \), \( p^x = (p_t^x)_{t \in \mathbb{R}_+} \) is a martingale on \( (\Omega, \mathbb{F}, \mathbb{P}) \).

▷ For every \( x \in E \) define
\[
\zeta^x := \inf \{ t \in \mathbb{R}_+ \mid p_t^x = 0 \}.
\]

▷ Let \( \Lambda^x := \{ \zeta^x < \infty, p_{\zeta^x}^x > 0 \} \in \mathcal{F}_{\zeta^x} \) and define
\[
\eta^x := \zeta^x \mathbb{I}_{\Lambda^x} + \infty \mathbb{I}_{\Omega \setminus \Lambda^x}, \quad x \in E
\]

Note that \( \eta^x \) (= **time at which** \( p^x \) **jumps to zero**) is a stopping time on \( (\Omega, \mathbb{F}) \).
Similar results (see) for the martingale deflators lead to:

**Theorem (one fixed $S$).** Let $\mathbb{P}[\eta^x < \infty, \Delta S_{\eta^x} \neq 0] = 0 \gamma$-a.e. If NA1($\mathbb{F}, S$) holds, then NA1($\mathbb{G}, S$) holds.

**Theorem (general stability).** TFAE:

1) $\eta^x = \infty \mathbb{P}$-a.s. for $\gamma$-a.e. $x \in E$.
2) for all $X \geq 0$ $\mathcal{B}_E \otimes \mathcal{O}(\mathbb{F})$-meas. s.t. $X^x$ $\mathbb{F}$-loc.martingale vanishing on $[\eta^x, \infty]$ $\gamma$-a.e., $X^{J/p^J}$ is a $\mathbb{G}$-loc.martingale
3) The process $1/p^J$ is a $\mathbb{G}$-loc.martingale

And 1) $\Rightarrow$ For any $S$ s.t. NA1($\mathbb{F}, S$) holds, NA1($\mathbb{G}, S$) also holds.

Some care for the converse (see); we can derive $\mathcal{H}'$-hyp. (see).
Example: failure of NA1($G, S^T$)

- Consider a Poisson($\lambda$) process $N$ stopped at time $T \in (0, \infty)$.
- Let $\mathcal{F}$ be the right-cont. filtration generated by $N$ and $J := N_T$.
- Then (Grorud,Pontier 2001) $p^x_T = e^{-\lambda T}x!/(\lambda T)^x \mathbb{I}_{\{N_T=x\}}$ and
  
  $p^x_t = e^{-\lambda t} \frac{\lambda(T-t)^{x-N_t} x!}{(\lambda T)^x (x-N_t)!} \mathbb{I}_{\{N_t \leq x\}}, \quad \forall \ t \in [0, T).$
Example: failure of $\text{NA1}(\mathcal{G}, S^T)$

- Consider a Poisson($\lambda$) process $N$ stopped at time $T \in (0, \infty)$.
- Let $\mathcal{F}$ be the right-cont. filtration generated by $N$ and $J := N_T$.
- Then (Grorud, Pontier 2001)
  \[ p_T^x = e^{-\lambda T} x^x/(\lambda T)^x \mathbb{I}_{\{N_T = x\}} \]
  \[ p_t^x = e^{-\lambda t} \frac{(\lambda(T-t))^{x-N_t} x!}{(\lambda T)^x(x-N_t)!} \mathbb{I}_{\{N_t \leq x\}}, \quad \forall \ t \in [0, T). \]

- $S_t := \exp(N_t - \lambda t(e-1))$, for all $t \in [0, T]$.
- $S$ is a strictly positive $\mathcal{F}$-martingale $\Rightarrow$ $\text{NA1}(\mathcal{F}, S)$ holds.
Example: failure of NA1($\mathbb{G}, S^T$)

- Consider a Poisson($\lambda$) process $N$ stopped at time $T \in (0, \infty)$.
- Let $\mathbb{F}$ be the right-cont. filtration generated by $N$ and $J := N_T$.
- Then (Grorud,Pontier 2001) $p_x^T = \frac{\lambda^T}{\lambda T} x!/(\lambda T)^x I\{N_T = x\}$ and
  $p_t^x = \frac{\lambda^T}{\lambda T} x!/(\lambda T)^x (x - N_t)! I\{N_t \leq x\}, \ \forall \ t \in [0, T]$.

- $S_t := \exp(N_t - \lambda t(e - 1))$, for all $t \in [0, T]$.
- $S$ is a strictly positive $\mathbb{F}$-martingale $\Rightarrow$ NA1($\mathbb{F}, S$) holds.

- Define the $\mathbb{G}$-stopping time $\sigma := \inf \{ t \in [0, T] | N_t = N_T \}$.

- For all $t \in [0, T]$, we get
  $(-I_{\sigma,T} \cdot S)_t = I\{t > \sigma\} \exp(N_\sigma - \lambda \sigma(e - 1)) \left(1 - \exp(-\lambda(t - \sigma)(e - 1))\right)$.

- $-I_{\sigma,T} \cdot S$ is nondecreasing, $\mathbb{P} [\sigma < T] = 1 \Rightarrow$ NA1($\mathbb{G}, S$) fails.
Example: failure of $\text{NA1}(\mathcal{G}, S^T)$

- Consider a Poisson($\lambda$) process $N$ stopped at time $T \in (0, \infty)$.
- Let $\mathcal{F}$ be the right-cont. filtration generated by $N$ and $J := N_T$.
- Then (Grorud, Pontier 2001) $p_T^x = e^{-\lambda T} x! / (\lambda T)^x \mathbb{I}_{\{N_T = x\}}$ and
  \[ p_t^x = e^{-\lambda t} \frac{(\lambda (T - t))^x - N_t}{(\lambda T)^x} \frac{x!}{(x - N_t)!} \mathbb{I}_{\{N_t \leq x\}}, \quad \forall \ t \in [0, T). \]

- $S_t := \exp(N_t - \lambda t(e - 1))$, for all $t \in [0, T]$.
- $S$ is a strictly positive $\mathcal{F}$-martingale $\Rightarrow$ NA1($\mathcal{F}, S$) holds.
- Define the $\mathcal{G}$-stopping time $\sigma := \inf \{ t \in [0, T] \mid N_t = N_T \}$.
- For all $t \in [0, T]$, we get
  \[ (-\mathbb{I}_{[\sigma, T]} \cdot S)_t = \mathbb{I}_{\{t > \sigma\}} \exp(N_{\sigma} - \lambda \sigma(e - 1)) \left(1 - \exp(-\lambda(t - \sigma)(e - 1))\right). \]
- $-\mathbb{I}_{[\sigma, T]} \cdot S$ is nondecreasing, $\mathbb{P} [\sigma < T] = 1 \Rightarrow$ NA1($\mathcal{G}, S$) fails.

Note: $p^x$ have positive probability to jump to zero exactly in correspondence of the jump times of the Poisson process $N$ (condition $\mathbb{P} [\eta^x < \infty, \Delta S_{\eta^x} \neq 0] = 0$ $\gamma$-a.e. fails).
An example with $\eta$ accessible

- Let $\zeta : \Omega \mapsto \mathbb{N}$ s.t. $p_k := \mathbb{P}[\zeta = k] \in (0, 1)$ $\forall k \in \mathbb{N}$, $\sum_k p_k = 1$.
- Set $\mathbb{F} = (\mathcal{F}_t)_{t \in \mathbb{R}_+}$ to be the smallest filtration that satisfies the usual hypotheses and makes $\zeta$ a stopping time.
- Since $\zeta$ is $\mathbb{N}$-valued, it is an accessible time on $(\Omega, \mathbb{F}, \mathbb{P})$.
- Define $\tau := \zeta - 1$.
- $Z_t = 0$ holds on $\{\zeta \leq t\}$. Moreover, with $q_k := \sum_{n=k+1}^{\infty} p_n$ $\forall k \in \{0, 1, \ldots\}$, and denoting $\lceil \cdot \rceil$ the integer part, on $\{t < \zeta\}$

$$Z_t = \mathbb{P}[\tau > t \mid \mathcal{F}_t] = \mathbb{P}[\zeta > t + 1 \mid \mathcal{F}_t] = \mathbb{P}[\zeta > \lceil t + 1 \rceil \mid \mathcal{F}_t] = \frac{q_{\lceil t+1 \rceil}}{q_{\lceil t \rceil}}.$$  

- $\zeta = \inf \{t \in \mathbb{R}_+ \mid Z_{t^-} = 0 \text{ or } Z_t = 0\}$.
- $Z_{\zeta^-} = q_{\lceil \zeta \rceil} / q_{\lceil \zeta - 1 \rceil} > 0$.
- $\eta = \zeta$; in particular, $\eta$ is accessible on $(\Omega, \mathbb{F}, \mathbb{P})$. 


Another example where NA fails and NA1 holds

\[ S_t = \exp \left( \sigma W_t - \frac{1}{2} \sigma^2 t \right). \]

For a given constant \( a \in (0, 1) \), define \( \tau := \sup \{ t : S_t = a \} \). Then

\[ Z_t := \mathbb{P} [ \tau > t | \mathcal{F}_t ] = \left( \frac{S_t}{a} \right)^{\wedge} 1 = \frac{N_t}{N^*_t}, \quad t \geq 0 \]

\[ N = \mathcal{E} \left( \frac{1}{a} \int \frac{1}{Z} 1_{\{ S < a \}} dS \right). \]

Note: \( \tau := \sup \{ t : N_t = N^*_\infty \} \).

\( \triangleright \) Since \( S \) is continuous, NA1(\( \mathbb{G}, S^\tau \)) holds.

\( \triangleright \) On the other hand, the following strategy realizes a classical arbitrage in the enlarged filtration at time \( \tau \) (see Aksamit et al.):

\[ \psi = \frac{1}{a} 1_{\{ S < a \}}. \]
Let $S = \mathcal{E}(\sigma W)$ and $\tau := \sup\{t \leq 1 : S_1 - 2S_t = 0\}$, that is, the last time before 1 when $S$ equals half of its value at time 1.

Here both $\text{NA}(\mathcal{G}, S^\tau)$ and $\text{NA1}(\mathcal{G}, S^\tau)$ hold $\Rightarrow \text{NFLVR}(\mathcal{G}, S^\tau)$. Indeed,

$$\{\tau \leq t\} = \left\{\inf_{t \leq s \leq 1} 2\frac{S_s}{S_t} \geq \frac{S_1}{S_t}\right\}.$$

Therefore,

$$\mathbb{P}[\tau \leq t | \mathcal{F}_t] = \mathbb{P}\left[\inf_{t \leq s \leq 1} 2S_{s-t} \geq S_{1-t}\right] = F(1 - t),$$

where $F(u) = \mathbb{P}[\inf_{s \leq u} 2S_s \geq S_u]$. Then $Z_t$ deterministic, decreasing $\Rightarrow \tau$ pseudo-stopping time and $S^\tau$ is a $\mathcal{G}$-martingale.

On the other hand:

after $\tau$ there are arbitrages and arbitrages of the first kind: at $\tau$ we know the value of $S_1$, and $S_t > S^\tau \forall t \in (\tau, 1]$. 
Local martingales in the initially enlarged filtration

For $x \in E$, $D^x$ denotes the $\mathbb{F}$-predictable compensator of $\mathbb{I}_{[\eta^x, \infty]}$.

**Lemma.** $D$ can be chosen $B_E \otimes \mathcal{P}(\mathbb{F})$-measurable and:

- $\Delta D^x < 1$ $\mathbb{P}$-a.s. ($\Rightarrow \mathcal{E}(-D^x) > 0$ and nonincreasing);
- $\mathcal{E}(-D^x)^{-1}\mathbb{I}_{[0, \eta^x]}$ is a $\mathbb{F}$-local martingale.

**Proposition.** Let $X \geq 0$ be $B_E \otimes \mathcal{O}(\mathbb{F})$-measurable, such that $X^x$ $\mathbb{F}$-local martingale vanishing on $[\eta^x, \infty]$ $\gamma$-a.e. Then $X^J/p^J$ is a $\mathcal{G}$-local martingale.

As an immediate consequence we have the following

**Key-Proposition.** Suppose there is $M \geq 0$, $B_E \otimes \mathcal{O}(\mathbb{F})$-measurable s.t. $M^x_0 = 1$, $M^x$ and $M^x S$ are $\mathbb{F}$-local martingales vanishing on $[\eta^x, \infty]$ $\gamma$-a.e. Then, $M^J/p^J$ is a $\mathcal{G}$-local martingale deflator.
Some converse implication

Recall that $D^x$ denotes the $\mathbb{F}$-predictable compensator of $\mathbb{I}_{[\eta^x, \infty]}$ and define $S^x := \mathcal{E}(-D^x)^{-1}\mathbb{I}_{[0, \eta^x]}$, $x \in E$.

**Theorem.** Let $\int_E \mathbb{P}[\eta^x < \infty] \gamma(dx) > 0$. Then NA1($\mathbb{F}$, $S^x$) holds for every $x \in E$, but NA1($\mathbb{G}$, $S^J$) fails.

Indeed, $S^x$ are $\mathbb{F}$-local martingales, $S^J = \mathcal{E}(-D^J)^{-1}$ is nondecreasing and $\mathbb{P}[S^J_t = S^J_0, \forall t \in \mathbb{R}_+] < 1$.

▶ An insider with knowledge of $J$ takes at time zero a position on a single unit of the stock with index $J$, and keeps it indefinitely. (The insider identifies from the beginning a single asset in the family $(S^x)_{x \in E}$ which will not default and can therefore arbitrage.)

Some particular cases depending on the law of $J$, see [here](#).

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Remarks

If $\sum_{k \in \mathbb{N}} \mathbb{P}[J = x_k] = 1$ holds for some family $(x_k)_{k \in \mathbb{N}} \subseteq E$,

- $\int_E \mathbb{P}[\eta^x < \infty] \gamma(dx) > 0 \Rightarrow \exists \kappa : \mathbb{P}[\eta^{x_\kappa} < \infty] > 0$;
- since $\mathbb{P}[\zeta^J < \infty] = 0$, then $\mathbb{P}[J = x_\kappa, \eta^{x_\kappa} < \infty] = 0$;
- the buy-and-hold strategy $\mathbb{I}\{J = x_\kappa\}$ in the single asset $S^{x_\kappa}$ results in the arbitrage $\mathbb{I}\{J = x_\kappa\} \cdot S^{x_\kappa}$.

(NA1($\mathbb{F}, S^{x_\kappa}$) holds while NA1($\mathbb{G}, S^{x_\kappa}$) fails)

If the law $\gamma$ has a diffuse component, one can still obtain an arbitrage of the first kind, under the stronger hypothesis:

- $\exists B \in \mathcal{B}_E$ with $\gamma[B] > 0$ s.t. $\mathbb{P}[\eta^B < \infty] > 0$, where $\eta^B$ is the time when the martingale $(\gamma_t[B])_{t \in \mathbb{R}_+}$ jumps to zero.

Indeed, denoting $D^B$ the $\mathbb{F}$-predictable compensator of $\mathbb{I}[\eta^B, \infty[$,

- $S := \mathcal{E}(-D^B)^{-1}\mathbb{I}[0, \eta^B]$ is a $\mathbb{F}$-local martingale, $\mathbb{I}\{J \in B\} \cdot S$ is nondecreasing, and $\mathbb{P}[S_t = S_0, \forall t \in \mathbb{R}_+] < 1$.

(NA1($\mathbb{F}, S$) holds while NA1($\mathbb{G}, S$) fails).
Proposition. Let $X \geq 0$ be $\mathcal{B}_E \otimes \mathcal{O}(\mathcal{F})$-measurable, such that $X^x$ \text{F-}supermartingale $\gamma$-a.e. Then $X^J/p^J$ is a $\mathcal{G}$-supermartingale.

(cf. concept of 'universal supermartingale density' in Imkeller, Perkowski 2013)

Remark. This can be used to establish that any semimartingale $X$ on $(\Omega, \mathcal{F}, \mathbb{P})$ remains a semimartingale on $(\Omega, \mathcal{G}, \mathbb{P})$. 

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